

ON MÜGER'S CENTRALIZER FOR A CERTAIN CLASS OF BRAIDED FUSION CATEGORIES

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ABSTRACT. We provide a general formula for Müger's centralizer of any fusion subcategory of a braided fusion category containing a tannakian subcategory. This entails a description for Müger's centralizer of all fusion subcategories of a group theoretical braided fusion category.

1. INTRODUCTION AND MAIN RESULTS

It is known that braided fusion categories containing a non-trivial Tannakian subcategory are equivariantizations of G -cross braided fusion categories [8, 9]. In this paper we present a formula for the Müger centralizer in equivariantizations of braided G -crossed fusion categories.

Let \mathcal{C} be a fusion category over k . The isomorphism class of an object X of \mathcal{C} will be denoted by $[X]$. However, when the context presents no ambiguity, we shall indicate an object and its class by the same letter. We shall use the notation $\text{Irr}(\mathcal{C})$ to indicate the set of isomorphism classes of simple objects of \mathcal{C} .

Let $t : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ be an action of G by tensor autoequivalences on a fusion category \mathcal{C} (see Section 1.1 for the precise definition). Such an action induces naturally an action of G on the set $\text{Irr}(\mathcal{C})$ of isomorphism classes of simple objects of \mathcal{C} .

A parameterization for all the fusion subcategories of an equivariantization will be given in [7]. For the sake of completeness of the paper we recall this description in Section 3. Below we recall the notion of a fusion datum which classifies all the fusion subcategories, see [7].

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1.1. Equivariantization under group actions. Let \mathcal{C} be a fusion category, G a finite group and $t : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ be an action of G on \mathcal{C} by k -linear tensor autoequivalences. Thus, for every $g \in G$, we have a k -linear tensor functor $t^g : \mathcal{C} \rightarrow \mathcal{C}$ and natural isomorphisms of tensors functors

$$t_2^{g,h} : t^g t^h \rightarrow t^{gh}, \quad g, h \in G,$$

and $t_0 : \text{id}_{\mathcal{C}} \rightarrow t^1$, subject to the following conditions:

$$(1.1) \quad (t_2^{ab,c})_X (t_2^{a,b})_{t^c(X)} = (t_2^{a,bc})_X t^a((t_2^{b,c})_X),$$

$$(1.2) \quad (t_2^{a,1})_X t_{t_0(X)}^a = (t_2^{1,a})_X (t_0)_{t^a(X)},$$

for all objects $X \in \mathcal{C}$, and for all $a, b, c \in G$. By the naturallity of $t_2^{g,h}$, $g, h \in G$, we have the following relation:

$$(1.3) \quad t^{gh}(f) (t_2^{g,h})_Y = (t_2^{g,h})_X t^g t^h(f),$$

for every morphism $f : Y \rightarrow X$ in \mathcal{C} . We shall assume in what follows that $t^1 = \text{id}_{\mathcal{C}}$ and $t_0, t_2^{g,1}, t_2^{1,g}$ are identities.

Let \mathcal{C}^G denote the corresponding *equivariantization*. Recall that \mathcal{C}^G is a finite semisimple k -linear category whose objects are G -equivariant objects of \mathcal{C} , that is, pairs (X, μ) , where X is an object of \mathcal{C} and $\mu = (\mu^g)_{g \in G}$, such that $\mu^g : t^g X \rightarrow X$ is an isomorphism, for all $g \in G$, satisfying

$$(1.4) \quad \mu^g t^g(\mu^h) = \mu^{gh} (t_2^{g,h})_X, \quad \forall g, h \in G, \quad \mu_e t_0 X = \text{id}_X.$$

A morphism $f : (X, \mu) \rightarrow (X', \mu')$ in \mathcal{C}^G is a morphism $f : X \rightarrow X'$ in \mathcal{C} such that $f \mu^g = \mu'^g$, for all $g \in G$.

We shall also say that an object X of \mathcal{C} is *G-equivariant* if there exists such a collection $\mu = (\mu^g)_{g \in G}$ so that $(X, \mu) \in \mathcal{C}^G$. Note that μ is not necessarily unique.

For all $g \in G$ the autoequivalence t^g is endowed with a monoidal structure $(t_2^g)_{X,Y} : t^g(X \otimes Y) \rightarrow t^g(X) \otimes t^g(Y)$, $X, Y \in \mathcal{C}$ such the following relation holds:

$$(1.5) \quad (t_2^{gh})_{X,Y} (t_2^{g,h})_{X \otimes Y} = ((t_2^{g,h^{-1}})_X \otimes (t_2^{g,h^{-1}})_Y) (t_2^g)_{t^h(X), t^h(Y)} t^g((t_2^h)_{X,Y}),$$

for all $g, h \in G$, $X, Y \in \mathcal{C}$.

Then \mathcal{C}^G is also a fusion category with tensor product $(X, \mu_X) \otimes (Y, \mu_Y) = (X \otimes Y, (\mu_X \otimes \mu_Y) t_{2X,Y})$, where for all $g \in G$, $(t_2^g)_{X,Y} : t^g(X \otimes Y) \rightarrow t^g(X) \otimes t^g(Y)$ is given by the monoidal structure on t^g .

The forgetful functor $F : \mathcal{C}^G \rightarrow \mathcal{C}$, $F(X, \mu) = X$, is a dominant functor. The functor F has a left adjoint $L : \mathcal{C} \rightarrow \mathcal{C}^G$, defined by $L(X) = (\oplus_{s \in G} t^s(X), \mu_X)$, where $(\mu_X)_g : \oplus_{t \in G} t^g t^s(X) \rightarrow \oplus_{s \in G} t^s(X)$ is given componentwise by the isomorphisms $(t_2^{g,s})_X$.

1.2. On the chosen set of isomorphisms C. Let Y be a simple object of \mathcal{C} and let $G_Y \subseteq G$ denote the inertia subgroup of Y , that is, $G_Y = \{g \in G \mid t^g(Y) \cong Y\}$. By the definition of G_Y , there exist isomorphisms $c_Y^g : t^g(Y) \rightarrow Y$, for all $g \in G_Y$. For all $g, h \in G_Y$, the composition $c_Y^g t^g(c_Y^h)(t_2^{g,h})^{-1}(c_Y^{gh})^{-1}$ defines an isomorphism $Y \rightarrow Y$. Since Y is a simple object, by Schur's Lemma, there exists a nonzero scalar $\alpha_Y(g, h) \in k^*$ such that

$$(1.6) \quad \alpha_Y(g, h)^{-1} \text{id}_Y = c_Y^g t^g(c_Y^h)(t_2^{g,h})^{-1}(c_Y^{gh})^{-1} : Y \rightarrow Y.$$

This defines a map $\alpha_Y : G_Y \times G_Y \rightarrow k^*$ which is a 2-cocycle on G_Y , see [6]. As explained in [6] the cohomology class of α_Y does not depend on the chosen set of isomorphisms c_Y^g with $g \in G_Y$. We denote by $\tilde{\alpha}_Y$ its cohomologous class.

Remark 1.1. The cocycle α_Y measures the possible obstruction for (Y, C_Y) to be a G_Y -equivariant object, where $u = (c_Y^g)_{g \in G_Y}$.

Consider another choice of isomorphisms $c_Y'^g : t^g(Y) \rightarrow Y$, $g \in G$. Then since Y is a simple object, the composition $c_Y^g(c_Y'^g)^{-1} : Y \rightarrow Y$ is given by scalar multiplication by some $f_Y(g) \in k^*$, for all $g \in G$. Denoting by α_Y' the 2-cocycle resulting from the choice of $(c_Y'^g)_{g \in G_Y}$, it is easy to see that α_Y and α_Y' differ by the coboundary of the cochain $f_Y : G_Y \rightarrow k^*$. This implies that the cohomology class $[\alpha_Y]$ of α_Y depends only on the isomorphism class of the simple object Y . More precisely one has the following relation (see [6]):

$$(1.7) \quad \alpha_Y'(g, h) = \frac{f_Y(gh)}{f_Y(g)f_Y(h)} \alpha_Y(g, h)$$

Fix a set $\{Y_0, Y_1, \dots, Y_s\}$ of representative objects for all the isomorphism classes of simple objects of \mathcal{C} . Denote $G_i := G_{Y_i}$ and let $c_{Y_i}^g : t^g(Y_i) \rightarrow Y_i$ be arbitrary isomorphisms with $c_{Y_i}^1 = \text{id}_{Y_i}$ for any $g \in G_i$. As above we define a 2-cocycle $\alpha_{Y_i} : G_i \times G_i \rightarrow k^*$ by the equality

$$(1.8) \quad \alpha_{Y_i}^{-1}(g, g') = c_{Y_i}^g t^g(c_{Y_i}^{g'})(t_2^{g,g'})^{-1}(c_{Y_i}^{gg'})^{-1}.$$

Note that $\alpha_{Y_i}(g, 1) = \alpha_{Y_i}(1, g) = 1$ for any $g \in G$, i.e. α_{Y_i} is a normalized cocycle. Moreover by Remark 7.1 one can assume that the values of $\alpha_{Y_i}(g, g')$ are all roots of unity.

For any simple object of \mathcal{C} with $Y \cong Y_i$ and $g \in G_Y$ we denote by $c_Y^g : T^g(Y) \rightarrow Y$ the isomorphism obtained by the composition $f^{-1}c_{Y_i}^g T^g(u)$ where $u : Y \rightarrow Y_i$ is an arbitrary chosen isomorphism. Note that c_Y^g does not depend on u and $c_Y^1 = \text{id}_Y$ for any simple object Y . Moreover $\alpha_Y(g, g') = \alpha_{Y_i}(g, g')$ for all $g, g' \in G_Y$. We denote this common value by $\alpha_{[Y]}(g, g')$. Moreover, since u is unique up to a scalar, the following diagram commutes:

$$\begin{array}{ccc} T^g(Y) & \xrightarrow{c_Y^g} & Y \\ T^g(v) \downarrow & & \downarrow v \\ T^g(Y_i) & \xrightarrow{c_{Y_i}^g} & Y_i \end{array}$$

for any $v : Y \rightarrow Y_i$ isomorphism.

For any two simple isomorphic objects $Y \cong Y'$ and any $f : Y \rightarrow Y'$ an isomorphism one has from above that

$$(1.9) \quad v \circ c_Y^g = c_{Y'}^g \circ t^g(v).$$

Indeed, since v is unique up to scalar, one can choose v to be composition of two isomorphisms $Y \rightarrow Y_i \rightarrow Y'$ where Y_i is the corresponding representative for Y and Y' . Then the commutativity follows from the definition of c_Y^g and commutativity of the above diagram.

Let $g \in G$ and Y be a simple object of \mathcal{C} . Since $t^g(Y)$ is also a simple object there is a scalar $D_{g,Y}(h) \in k^*$ such that

$$(1.10) \quad c_{t^g(Y)}^{ghg^{-1}} = D_{g,Y}^{-1}(h) [t^g(c_Y^h)(t_2^{g,h})_Y^{-1}(t_2^{ghg^{-1},g})_Y]$$

for any $h \in G_Y$. From the commutative diagram below it is easy to see that this scalar does not depend on the class of isomorphism of Y .

$$\begin{array}{ccccccccc} t^{ghg^{-1}}(t^g(Y)) & \xrightarrow{(t_2^{ghg^{-1},g})_Y} & t^{gh}(Y) & \xrightarrow{(t_2^{g,h})_Y^{-1}} & t^g(t^h(Y)) & \xrightarrow{t^g(c_Y^h)} & t^g(Y) & \xrightarrow{[c_{t^g(Y)}^{ghg^{-1}}]^{-1}} & t^{ghg^{-1}}(t^g(Y)) \\ \downarrow t^{ghg^{-1}}(t^g(f)) & & \downarrow t^{gh}(f) & & \downarrow t^g(t^h(f)) & & \downarrow t^g(f) & & \downarrow t^{ghg^{-1}}(f) \\ t^{ghg^{-1}}(t^g(Y_i)) & \xrightarrow{(t_2^{ghg^{-1},g})_{Y_i}} & t^{gh}(Y_i) & \xrightarrow{(t_2^{g,h})_{Y_i}^{-1}} & t^g(t^h(Y_i)) & \xrightarrow{t^g(c_{Y_i}^h)} & t^g(Y_i) & \xrightarrow{[c_{t^g(Y_i)}^{ghg^{-1}}]^{-1}} & t^{ghg^{-1}}(t^g(Y_i)) \end{array}$$

Note that the last rectangle is commutative by Equation (1.9). The other rectangles are commutative by the naturality of the group action.

We denote this common value by $D_{g,[Y]}(h)$.

Denote by $C_Y := \{c_Y^g\}_{g \in G_Y}$ the set of chosen isomorphisms for any simple object Y of \mathcal{C} . One can choose C_Y such that $c_Y^1 = \text{id}_Y$. In this case one has that $\alpha_Y(1, h) = \alpha_Y(h, 1) = 1$. *For the rest of the paper we fix a set $C := \cup_{Y \in \text{Irr}(\mathcal{C})} C_Y$ of chosen isomorphisms as above. Moreover note that we can choose $c_1^g = \text{id}_1$ for any $g \in G$ since $t^1 = \text{id}_{\mathcal{C}}$ and t_0 is assumed to be identity.*

1.3. Hom spaces as projective representations. Let $(X, \mu) \in \mathcal{C}^G$ and let $Y \in \text{Irr}(\mathcal{C})$. Then following [6] the space $\text{Hom}_{\mathcal{C}}(Y, X)$ becomes a projective representation of G_Y with factor set α_Y . Its structure is defined by

$$(1.11) \quad \pi(g)(f) = \mu^g t^g(f)(c_Y^g)^{-1} : Y \rightarrow X,$$

for all $f \in \text{Hom}_{\mathcal{C}}(Y, N)$.

Let $U, Y, Z \in \text{Irr}(\mathcal{C})$. Recall by [6, Proposition 3.6] that the vector space $\text{Hom}_{\mathcal{C}}(U, t^m(Y) \otimes t^n(Z))$ is a projective representation of $T := G_U \cap G_{t^m(Y)} \cap G_{t^n(Z)}$ with associated 2-cocycle $(\alpha_{[U]}|T)(\alpha_{[Y]}|T)^{-1}(\alpha_{[Z]}|T)^{-1}$. The action of $g \in T$ is given by

$$(1.12) \quad g \cdot f = (t^m(c_Y^{m^{-1}gm}) \otimes t^n(c_Z^{n^{-1}gn}))((t^2)_Y^{m, m^{-1}gm} \otimes (t^2)_Z^{n, n^{-1}gn})(t_2^g)^{t^m(Y), t^n(Z)} t^g(f)(c_U^g)^{-1}$$

for all $f \in \text{Hom}_{\mathcal{C}}(U, t^m(Y) \otimes t^n(Z))$.

Note that the definition of the above projective representation depend on all chosen set of isomorphisms c_Y, c_Z, c_U .

As in [6] denote this projective representation by $\tau_U^{Y,Z}(m, n)$. For shortness we write $\tau_U^{Y,Z} := \tau_U^{Y,Z}(1, 1)$.

Note that $\tau_U^{Y,Z}(m, n)$ depend on the chosen set of isomorphisms for $U, t^m(Y), t^n(Z)$. Clearly changing these system C of isomorphisms one obtains an equivalent projective representation.

However, note that for the above system C of chosen isomorphisms one has that $\tau_U^{Y,Z}(m, n) \simeq \tau_{Y_i}^{Y_j, Y_k}(m, n)$ as T -projective representations, where $U \cong Y_i$, $Y \cong Y_j$ and $Z \cong Y_k$. Indeed, choosing isomorphisms $u : U \rightarrow Y_i$, $v : Y \rightarrow Y_j$ and $w : Z \rightarrow Y_k$ it is easy to verify (by the naturality of the G -action) that the k -linear map $\phi : \text{Hom}_{\mathcal{C}}(U, t^m(Y) \otimes t^n(Z)) \rightarrow \text{Hom}_{\mathcal{C}}(Y_i, t^m(Y_j) \otimes t^n(Y_k))$ given by $f \mapsto u^{-1}f(t^m(v) \otimes t^n(w))$ is an isomorphism of projective representations. This is equivalent to the commutativity of the diagram below. In this diagram the first and the last rectangles are commutative by Equation (1.9). The other rectangles are commutative by the naturality of the action of G .

$$\begin{array}{ccc}
U & \xrightarrow{u} & Y_i \\
\downarrow (c_U^g)^{-1} & & \downarrow (c_{Y_i}^g)^{-1} \\
t^g(U) & \xrightarrow{t^g(u)} & t^g(Y_i) \\
\downarrow t^g(f) & & \downarrow t^g(\phi(f)) \\
t^g(t^m(Y) \otimes t^n(Z)) & \xrightarrow{t^g(t^m(v) \otimes t^n(w))} & t^g(t^m(Y_j) \otimes t^n(Y_k)) \\
\downarrow (t^g)_2^{t^m(Y), t^n(Z)} & & \downarrow (t^g)_2^{t^m(Y_j), t^n(Y_k)} \\
t^g(t^m(Y)) \otimes t^g(t^n(Z)) & \xrightarrow{t^g(t^m(v)) \otimes t^g(t^n(w))} & t^g(t^m(Y_j)) \otimes t^g(t^n(Y_k)) \\
\downarrow (t_2)_Y^{g,m} \otimes (t_2)_Z^{g,n} & & \downarrow (t_2)_{Y_j}^{g,m} \otimes (t_2)_{Y_k}^{g,n} \\
t^{gm}(Y) \otimes t^{gn}(Z) & \xrightarrow{t^{gm}(v) \otimes t^{gn}(w)} & t^{gm}(Y_j) \otimes t^{gn}(Y_k) \\
\downarrow (t_2^{m,m^{-1}gm})_Y^{-1} \otimes (t_2^{n,n^{-1}gn})_Z^{-1} & & \downarrow (t_2^{m,m^{-1}gm})_{Y_j}^{-1} \otimes (t_2^{n,n^{-1}gn})_{Y_k}^{-1} \\
t^m(t^{m^{-1}gm}(Y)) \otimes t^n(t^{n^{-1}gn}(Z)) & \xrightarrow{t^m(t^{m^{-1}gm}(v)) \otimes t^n(t^{n^{-1}gn}(w))} & t^m(t^{m^{-1}gm}(Y_j)) \otimes t^n(t^{n^{-1}gn}(Y_k)) \\
\downarrow t^m(c_Y^{m^{-1}gm}) \otimes t^n(c_Z^{n^{-1}gn}) & & \downarrow t^m(c_{Y_j}^{m^{-1}gm}) \otimes t^n(c_{Y_k}^{n^{-1}gn}) \\
t^m(Y) \otimes t^n(Z) & \xrightarrow{t^m(v) \otimes t^n(w)} & t^m(Y_j) \otimes t^n(Y_k)
\end{array}$$

For this reason, by fixing the system C as above for the rest of the paper, we use the notation $\chi_{\tau_{[U]}^{[Y],[Z]}(m,n)}$ for the character of these isomorphic representations. By abuse of notation sometimes we also write $\chi_{\tau_U^{Y,Z}(m,n)}$ instead of $\chi_{\tau_{[U]}^{[Y],[Z]}(m,n)}$.

1.4. Fusion datum. In this subsection we recall the notion of a fusion datum relative to a given set of chosen isomorphisms C .

Definition 1.2. A C -fusion datum with respect to the tensor auto equivalence action $\mathfrak{t} : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ is a triple $(\mathcal{S}, H, \lambda)$ where:

- (1) \mathcal{S} is a G -stable fusion subcategory of \mathcal{C} .
- (2) H is a normal subgroup of G acting trivially on $\text{Irr}(\mathcal{S})$.
- (3) $\alpha_Y|_H$ is cohomologous trivial for any simple object Y of \mathcal{S} . Or equivalently the object Y is H -equivariant.
- (4) Every element $h \in H$ acts as a scalar on the projective representation $\tau_U^{Y,Z}(t, s)$, for any simple objects $Y, Z, U \in \text{Irr}(\mathcal{S})$ and any $t, s \in G$ such that U is a constituent of $\mathfrak{t}^t(Y) \otimes \mathfrak{t}^s(Z)$.
- (5) $\lambda : \text{Irr}(\mathcal{S}) \times H \rightarrow k^*$ is a map satisfying the following conditions:
 - (i) The following holds, for all $[Y] \in \text{Irr}(\mathcal{S})$, $h, h' \in H$:

$$(1.13) \quad \lambda([Y], hh') = \lambda([Y], h) \lambda([Y], h') \alpha_{[Y]}(h, h').$$

- (ii) For all $h \in H$ the map $\lambda(-, h) : \text{Irr}(\mathcal{S}) \rightarrow U(1)$ is a *twisted grading* on \mathcal{S} with respect to τ , that is,

$$(1.14) \quad \lambda([U], h) = \lambda([Y], h) \lambda([Z], h) \left(\frac{\chi_{\tau_{[U]}^{[Y],[Z]}}(h)}{\chi_{\tau_{[U]}^{[Y],[Z]}}(1)} \right)$$

for any irreducible constituent U of $Y \otimes Z$.

- (iii) λ is a G -invariant function, i.e:

$$(1.15) \quad \lambda([\mathfrak{t}^g(Y)], h) = \lambda([Y], g^{-1}hg) D_{g, [Y]}(g^{-1}hg),$$

for all $g \in G$, $h \in H$, $Y \in \text{Irr}(\mathcal{S})$.

Note that Condition (2) from above can be expressed as $H \subseteq \bigcap_{Y \in \text{Irr}(\mathcal{S})} G_Y$, where G_Y is the inertia group of Y .

Then a parameterization of fusion subcategories of an equivariantization can be given in terms of the *fusion data* introduced above:

Theorem 1.3. ([7]) *Let \mathcal{C} be a fusion category and let $\mathfrak{t} : \underline{G} \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ be an action of a finite group G on \mathcal{C} by tensor autoequivalences. Fix a system C of chosen isomorphism as above. Then there is a bijective correspondence between the following two sets:*

- (a) *Fusion subcategories $\mathcal{D} \subseteq \mathcal{C}$.*
- (b) *C -fusion data $(\mathcal{S}, H, \lambda)$.*

We denote by $\mathcal{C}(\mathcal{S}, H, \lambda)$ the category associated to the C -fusion datum $(\mathcal{S}, H, \lambda)$.

For the sake of completeness we will recall the proof of Theorem 1.3 in Section 3. In order to establish Theorem 1.3, we use the explicit description of all simple objects of the equivariantization \mathcal{C}^G from [6]. The isomorphism classes of simple objects of \mathcal{C}^G are parameterized by pairs $([Y], \pi)$, where $[Y]$ runs over the orbits of the action of G on the isomorphism classes of simple objects of \mathcal{C} , and π is an irreducible α_Y -projective representation of the inertia subgroup $G_{[Y]}$ ([6]). The irreducible projective representation π has an associated two cocycle $\alpha_Y \in Z^2(G_{[Y]}, k^*)$ (see Equation 1.6). The corresponding simple object associated to the pair $([Y], \pi)$ is denoted by $S_{Y,\pi}$.

1.5. G -cross braided fusion categories. It is known that braided fusion categories containing a non-trivial Tannakian subcategory are equivariantizations of G -cross braided fusion categories [9, 8].

In the second part of the paper we consider equivariantizations of braided G -crossed fusion categories. For a braided G -crossed category we define a notion of a G -centralizer (see subsection 5.4). We say that two simple objects X, Y of a braided G -crossed category \mathcal{C} G -centralize each other if there are two simple objects of the equivariantization \mathcal{C}^G which centralize each other and have X respectively Y as constituents. For two fusion categories \mathcal{D}, \mathcal{E} of \mathcal{C} we write $\mathcal{D} \perp_G \mathcal{E}$ if any two objects of \mathcal{D} and \mathcal{E} G -centralize each other

Let $(\mathcal{S}, H, \lambda)$ be a C-fusion datum and suppose that the support of \mathcal{S} is a subgroup $K \leq G$. In this part of the paper we adopt the notation $\mathcal{D}(H, K, \mathcal{S}, \lambda)$ for the fusion subcategory $\mathcal{D}(\mathcal{S}, H, \lambda)$ associated to the given fusion datum. Recall that if $\mathcal{D} \subset \mathcal{C}$ is a fusion subcategory of a graded fusion category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ then its support is defined by

$$(1.16) \quad \text{supp}(\mathcal{D}) = \{g \in G \mid \mathcal{D} \cap \mathcal{C}_g \neq 0\}.$$

In the next theorem we give a description for the fusion datum associated to the Müger centralizer of a given fusion subcategory.

Theorem 1.4. *Let \mathcal{C} be a braided G -crossed fusion category and $(\mathcal{S}, H, \lambda)$ a C-fusion datum with $K := \text{supp}(\mathcal{S})$. Then the following hold:*

- (1) $[K, H] = 1$, where $[x, y] = xyx^{-1}y^{-1}$ is the usual commutator.
- (2) *One has that:*

$$\mathcal{C}(H, K, \mathcal{S}, \lambda)' = \mathcal{C}(K, H', \mathcal{T}, \tilde{\lambda})$$

where \mathcal{T} is a fusion subcategory of \mathcal{C} whose support is $H' \subseteq H$. Moreover \mathcal{T} is stable under the action of G and it satisfies $\mathcal{S} \perp_G \mathcal{T}$.

- (3) *With the above notations if $a \in K$ and $b \in H'$ then one has that $\tilde{\lambda} : \text{Irr}(\mathcal{T}) \times K \rightarrow k^*$ is given by*

$$(1.17) \quad \tilde{\lambda}([Y_b], a) = \lambda^{-1}([X_a], b) \omega([X_a], [Y_b])$$

for any object $X_a \in \mathcal{C}_a \cap \mathcal{S}$ and any $Y_b \in \mathcal{C}_b \cap \mathcal{T}$.

- (4) *Moreover if \mathcal{C}^G is nondegenerate then $H' = K$.*

The roots of unity $\omega([X_a], [Y_b])$ are defined in subsection 5.5. Note that in the nondegenerate case one obtains that

$$\mathcal{C}(H, K, \mathcal{S}, \lambda)' = \mathcal{C}(K, H, \mathcal{T}, \tilde{\lambda})$$

formula that generalizes [16, Proposition 3.6] and [16, Proposition 5.2] to any nondegenerate equivariantization of a braided G -crossed fusion category.

Recall that [17, Theorem 5.3] states that every braided group-theoretical category is equivalent to $C(\xi)^G$, for some pointed group crossed category $C(\xi)$ associated to a normalized quasi-abelian 3-cocycle ψ on a finite crossed module (G, X, δ) . Thus it follows from Theorem 1.4 that we can give a description for the Müger centralizer of any fusion subcategory of all braided group theoretical fusion categories.

1.6. Organization of the paper. The paper is organized as follows. In the first section we recall the description of the simple objects $S_{Y,\pi} \in \mathcal{C}^G$ from [6]. The fusion rules of \mathcal{C}^G from loc. cit. is also recalled in this section. Next section recalls a proof of Theorem 1.3. In Proposition 3.10 we show that any fusion subcategory of \mathcal{C}^G is also an equivariantization of a fusion subcategory of \mathcal{C} by a quotient subgroup of G . In this section we also give a description for the lattice of fusion subcategories of an equivariantization \mathcal{C}^G . In Section 4 we recall basic notions on braided G -cross fusion categories. In Section 5 we study equivariantizations of braided G -crossed fusion categories and describe Müger's centralizer for a fusion subcategory. Section 6 contains some examples and application of our results.

2. SIMPLE OBJECTS OF AN EQUIVARIANTIZATION

In this section we recall the description of the simple objects of an equivariantization \mathcal{C}^G as well as the fusion rules for the fusion category \mathcal{C}^G which were described in [6].

2.1. The relative adjoint functor. For any simple object Y consider the forgetful functor $F_Y : \mathcal{C}^G \rightarrow \mathcal{C}^{G_Y}$. Let \mathcal{R} be a set of representatives of the left cosets of G_Y in G . Thus $G = \cup_{t \in \mathcal{R}} tG_Y$.

For all $(N, \nu) \in \mathcal{C}^{G_Y}$ let $L_Y^{\mathcal{R}}(N, \nu) = (\oplus_{t \in \mathcal{R}} t^t(N), \mu) \in \mathcal{C}^G$, where, for all $g \in G$, the equivariant structure $\mu_{L_Y^{\mathcal{R}}(N, \nu)}^g : \oplus_{t \in \mathcal{R}} t^{gt}(N) \rightarrow \oplus_{t \in \mathcal{R}} t^t(N)$ is defined componentwise by the formula

$$(2.1) \quad \mu_{L_Y^{\mathcal{R}}(N, \nu)}^{g,t} = t^s(\nu^h)(t_2^{s,h})^{-1} t_2^{g,t} : t^g t^t(N) \rightarrow t^s(N),$$

and the elements $h \in G_Y$, $s \in \mathcal{R}$ are uniquely determined by the relation

$$(2.2) \quad gt = sh.$$

It was noticed in [6] that if $(N, \nu) \in \mathcal{C}^{G_Y}$ then $L_Y(N, \nu) \in \mathcal{C}^G$. Therefore there is a well defined functor $L_Y^{\mathcal{R}} : \mathcal{C}^{G_Y} \rightarrow \mathcal{C}^G$ which by results of *loc.cit* is left adjoint of the forgetful functor F_Y . By the uniqueness of the adjoint functor, it follows that, up to isomorphism, $L_Y^{\mathcal{R}}$ does not depend on the particular set choice of the set of representatives \mathcal{R} . For this reason we will use the notation L_Y instead of $L_Y^{\mathcal{R}}$.

2.1.1. Notations. Let π be a α -projective representation of a finite group G . We usually denote by V_π the vector space associated to the projective representation π of G . Therefore one can regard π as a map $\pi : G \rightarrow \text{End}_k(V_\pi)$ with $\pi(gg') = \alpha(g, g')\pi(g)\pi(g')$ for all $g, g' \in G$. We also denote by χ_π the character of the projective representation π , thus $\chi_\pi(h) = \text{tr}_{V_\pi}(L_h)$.

2.1.2. Simple objects of \mathcal{C}^{G_Y} . Recall [6] that to any $\alpha_{[Y]}$ -projective representation π of G_Y one can associate an object $\pi \otimes Y \in \mathcal{C}^{G_Y}$. As an object of \mathcal{C} we have that $\pi \otimes Y$ coincides to $V_\pi \otimes Y$ and its equivariant structure ν^g is defined as follows:

$$(2.3) \quad \nu^g : t^g(V_\pi \otimes Y) = V_\pi \otimes t^g(Y) \xrightarrow{\pi(g) \otimes c_Y^g} V_\pi \otimes Y.$$

for all $g \in G_Y$. Conversely, if $L = (N, \nu)$ is any object of \mathcal{C}^{G_Y} with $N \simeq \text{Hom}_{\mathcal{C}}(Y, N) \otimes Y$, then $V = \text{Hom}_{\mathcal{C}}(Y, N)$ carries a projective representation π of G_Y with the factor set $\tilde{\alpha}_Y$. This projective representation is defined as in (1.11).

In other words, there is a bijective correspondence between isomorphism classes of simple objects $L = (N, \nu)$ of \mathcal{C}^{G_Y} such that $N \simeq \text{Hom}_{\mathcal{C}}(Y, N) \otimes Y$ and irreducible projective representations of the group G_Y with factor set $\tilde{\alpha}_Y$. If the simple object L corresponds to the projective representation π , we have $\text{FPdim } L = \dim \pi \text{ FPdim } Y$.

Note that the simple object $Y \otimes V_\pi$ depends on the chosen set of isomorphisms C_Y at Y .

2.1.3. *Parameterization of simple objects.* In [6, Theorem 2.12] it was shown that there is a bijective correspondence between the set isomorphism classes of simple objects (X, μ) of \mathcal{C}^G and the set of isomorphism classes of pairs (Y, π) , where Y runs over the orbits of the action of G on $\text{Irr}(\mathcal{C})$ and π is an irreducible α_Y -projective representation of the inertia subgroup $G_Y \subseteq G$.

The simple object corresponding to the pair (Y, π) was denoted by $S_{Y,\pi}^{C_Y}$ in [6]. Then $S_{Y,\pi}^{C_Y} := L_Y(Y \otimes \pi)$ where $Y \otimes V_\pi \in \mathcal{C}^{G_Y}$ was previously defined. Thus we have $S_{Y,\pi} \simeq \oplus_{t \in G/G_Y} t^t(V_\pi \otimes Y)$ and its equivariant structure follows from Equation (2.1). In particular, one has that $\text{FPdim } S_{Y,\pi} = \dim \pi [G : G_Y] \text{FPdim } Y$.

Remark 2.1. Note that $S_{Y,\pi}^{C_Y}$ as defined above depends on the set of chosen isomorphisms C_Y defined previously. Let us describe more explicitly the dependence of the simple object $S_{Y,\pi}^{C_Y}$ on the choice of the isomorphisms $c_Y^g : t^g(Y) \rightarrow Y$. Suppose we choose another set of isomorphisms $c'_Y = \{c_Y'^g\}$. Then, Y being a simple object, implies that for any $g \in G_Y$ one can write that $c_Y'^g = f_Y^{C,C'}(g)c_Y^g$, for some scalar $f_Y^{C,C'}(g) \in k^*$. It follows easily that $\pi \otimes Y = (f_Y^{C,C'})^{-1}\pi \otimes Y$ as objects in \mathcal{C}^{G_Y} . Hence

$$(2.4) \quad S_{Y,\pi}^{C_Y} = L_Y(\pi \otimes Y) = L_Y((f_Y^{C,C'})^{-1}\pi \otimes Y) = S_{Y,(f_Y^{C,C'})^{-1}\pi}^{C'_Y}.$$

Remark 2.2. One also can check that if $Y \simeq Y'$ and $c_{Y'}$ is obtained from c_Y via an isomorphism $f : Y \rightarrow Y'$ (i.e. satisfying Equation (1.9)) then $Y \otimes V_\pi \cong Y' \otimes V_\pi$ as objects of \mathcal{C}^{G_Y} . Therefore in this case $S_{Y,\pi}^{c_Y} \simeq S_{Y',\pi}^{c_{Y'}}$.

For the rest of this paper we write for shortness $S_{Y,\pi}$ for $S_{Y,\pi}^{C_Y}$ where $C_Y \subset \mathcal{C}$ is the fixed set of chosen isomorphisms defined in the previous section.

2.2. On the choice of the isomorphisms on the orbit. Let G be a finite group and π be a α -projective representation of a subgroup $H \leq G$. For any $g \in G$ define ${}^g\pi$ as a projective representation of ${}^gH := gHg^{-1}$ by ${}^g\pi(ghg^{-1}) := \pi(h)$ for $h \in H$. Note that ${}^g\pi$ is a ${}^g\alpha$ -projective representation of gH where ${}^g\alpha(ghg^{-1}, gh'g^{-1}) = \alpha(h, h')$.

Similarly, for any function $f : G_Y \rightarrow k$ define by ${}^gf : G_{t^g(Y)} \rightarrow k$ defined by ${}^gf(ghg^{-1}) := f(h)$ for all $h \in G_Y$.

With the notations from the previous section let $g \in G_Y$. Since $t^g(Y)$ is a constituent of $F_Y(S_{Y,\pi})$, it follows from [6, Theorem 2.2] that $S_{Y,\pi} =$

$S_{t^g(Y), \delta}$, for some irreducible projective representation δ of $G_{t^g(Y)}$. In this section we discuss the dependence of δ upon π . Recall that by Equation (1.10) one has

$$(2.5) \quad t^g(c_Y^h)(t_2^{g,h})_Y^{-1}(t_2^{ghg^{-1},g})_Y = D_{g,[Y]}(h)c_{t^g(Y)}^{ghg^{-1}}$$

for any $h \in G_Y$.

Lemma 2.3. *With the above notations one has*

$$(2.6) \quad S_{Y,\pi} \cong S_{t^g(Y), {}^gD_{g,[Y]}(-) {}^g\pi},$$

for all $g \in G$.

Proof. Recall that $S_{Y,\pi} = L_Y(\pi \otimes Y)$. As in the proof of [6, Theorem 2.2] it is enough to check that $V_{{}^gD_{g,[Y]}(-) {}^g\pi} \otimes t^g(Y)$ is a constituent of the restriction $F_{t^g(Y)}(L_Y(\pi \otimes Y))$ where $F_{t^g(Y)}$ is the forgetful functor $F_{t^g(Y)} : \mathcal{C}^G \rightarrow \mathcal{C}^{G_{t^g(Y)}}$. As objects of \mathcal{C} , $F_{t^g(Y)}(L_Y(\pi \otimes Y)) = \bigoplus_{s \in G/G_Y} t^s(V_\pi \otimes Y)$. Restricting the equivariant structure $\mu_{S_{Y,\pi}}$ of $S_{Y,\pi}$ to an element $ghg^{-1} \in G_{t^g(Y)}$ it follows that $(ghg^{-1})g = gh$, and therefore one can choose $s = g$, with notations as in Equation (2.2). Thus on the component $t^g(Y) \otimes V_\pi$ the equivariant structure $\mu_{S_{Y,\pi}}^{ghg^{-1}}$ is given by the composition

$$\begin{aligned} t^{ghg^{-1}}(V_\pi \otimes t^g(Y)) &\xrightarrow{\text{id}_{V_\pi} \otimes (t_2^{tht^{-1},t})_Y} V_\pi \otimes t^{gh}(Y) \rightarrow \\ &\xrightarrow{\text{id}_{V_\pi} \otimes (t_2^{t,h})_Y^{-1}} V_\pi \otimes t^g(t^h(Y)) \xrightarrow{\pi(h) \otimes t^g(c_Y^h)} V_\pi \otimes t^g(Y). \end{aligned}$$

This can also be written as

$$V_\pi \otimes t^{ghg^{-1}}(t^g(Y)) \xrightarrow{\pi(h) \otimes (t^g(c_Y^h)(t_2^{g,h})_Y(t_2^{ghg^{-1},g})_Y)} V_\pi \otimes t^g(Y).$$

Formula (2.5) shows that the above morphism coincides with the equivariant structure on ${}^gD_Y(g, -) {}^g\pi \otimes t^g(Y)$ given by Equation (2.3). \square

Remark 2.4. With the above notations one can check by straightforward computation that

$$t^g(Y \otimes V_\pi) = t^g(Y) \otimes {}^gD_{g,[Y]}(-) {}^g\pi$$

as equivariant objects of $\mathcal{C}^{G_{t^g(Y)}}$. This also implies that

$$\begin{aligned} S_{Y,\pi} = L_Y(\pi \otimes Y) &\simeq L_{t^g(Y)}(t^g(\pi \otimes Y)) \simeq \\ &\simeq L_{t^g(Y)}(t^g(Y) \otimes {}^gD_{g,[Y]}(-) {}^g\pi) = S_{t^g(Y), {}^gD_{g,[Y]}(-) {}^g\pi} \end{aligned}$$

It can also shown by straightforward computation that

$$(2.7) \quad \alpha_{[Y]}(h, h') = \frac{D_{g,[Y]}(hh')}{D_{g,[Y]}(h)D_{g,[Y]}(h')} \alpha_{[t^g(Y)]}(ghg^{-1}, gh'g^{-1}),$$

for all $h, h' \in G_Y$, $g \in G$.

2.3. More on the projective representations τ . Recall from the previous section that we denoted by $\tau_U^{Y,Z} := \tau_U^{Y,Z}(1, 1)$.

Proposition 2.5. *With the above notations it follows that*

$$\tau_{[U]}^{[t^m(Y)], [t^n(Z)]} = \tau_{[U]}^{[Y], [Z]}(m, n) {}^m(D_{m, [Y]})^{-1} {}^n(D_{n, [Z]})^{-1}$$

i.e for any $g \in G_U \cap mG_Y m^{-1} \cap nG_Z n^{-1}$ one has:

$$(2.8) \quad \tau_{[U]}^{[t^m(Y)], [t^n(Z)]}(g) = \tau_{[U]}^{[Y], [Z]}(m, n)(g) D_{m, [Y]}^{-1}(m^{-1}gm) D_{n, [Z]}^{-1}(n^{-1}gn)$$

Proof. Note that both projective representations are defined on the same vector space $\text{Hom}_{\mathcal{C}}(U, t^m(Y) \otimes t^n(Z))$. Moreover, for any $g \in G_U \cap mG_Y m^{-1} \cap nG_Z n^{-1}$ and $f \in \text{Hom}_{\mathcal{C}}(U, t^m(Y) \otimes t^n(Z))$ using Equation (2.5) one has the following

$$[\tau_{[U]}^{[Y], [Z]}(m, n)](g)f = (c_{t^m(Y)}^g \otimes c_{t^n(Z)}^g)(t^g)_2^{t^m(Y), t^n(Z)} t^g(f)(c_U^g)^{-1}.$$

Similarly one has that

$$\begin{aligned} [\tau_{[U]}^{[t^m(Y)], [t^n(Z)]}](g)f &= [t^m(c_Y^{m^{-1}gm}) \otimes t^n(c_Z^{n^{-1}gn})][(t_2^{m, m^{-1}gm})_Y^{-1} \otimes (t_2^{n, n^{-1}gn})_Z^{-1}] \\ &\quad [(t_2^{g, m})_Y \otimes (t_2^{g, n})_Z] [(t^g)_2^{t^m(Y), t^n(Z)} t^g(f)(c_U^g)^{-1}] \\ &= [(t^m(c_Y^{m^{-1}gm})(t_2^{m, m^{-1}gm})_Y^{-1} (t_2^{g, m})_Y) \otimes [t^n(c_Z^{n^{-1}gn})(t_2^{n, n^{-1}gn})_Z^{-1} (t_2^{g, n})_Z]]. \end{aligned}$$

On the other hand, by Equation (1.10) note that

$$c_{t^m(Y)}^g = c_{t^m(Y)}^{m(m^{-1}gm)m^{-1}} = D_{m, Y}^{-1}(m^{-1}gm)[t^m(c_Y^{m^{-1}gm})(t_{m, m^{-1}gm}^2)_Y^{-1}(t_{g, m}^2)_Y].$$

Similarly one has that

$$c_{t^n(Z)}^g = c_{t^n(Z)}^{n(n^{-1}gn)n^{-1}} = D_{n, Z}^{-1}(n^{-1}gn)[t^n(c_Z^{n^{-1}gn})(t_{n, n^{-1}gn}^2)_Z^{-1}(t_{g, n}^2)_Z]$$

Thus

$$\begin{aligned} [\tau_{[U]}^{[Y], [Z]}(m, n)](g)f &= D_{m, [Y]}^{-1}(m^{-1}gm) D_{n, [Z]}^{-1}(n^{-1}gn) \\ &\quad ([t^m(c_Y^{m^{-1}gm})(t_2^{m, m^{-1}gm})_Y^{-1} (t_2^{g, m})_Y] \otimes [t^n(c_Z^{n^{-1}gn})(t_2^{n, n^{-1}gn})_Z^{-1} (t_2^{g, n})_Z]) \\ &\quad (t^g)_2^{t^m(Y), t^n(Z)} t^g(f)(c_U^g)^{-1} \\ &= D_{m, [Y]}^{-1}(m^{-1}gm) D_{n, [Z]}^{-1}(n^{-1}gn) \tau_{[U]}^{[t^m(Y)], [t^n(Z)]}(g)f. \end{aligned}$$

□

2.4. On the tensor product of two simple objects. Let $S_{Y, \pi}$ and $S_{Z, \gamma}$ be two simple objects of \mathcal{C}^G . The following Corollary follows from the proof of [6, Theorem 3.9], see also [6, Corollary 3.6].

Corollary 2.6. *A simple object $S_{U, \delta}$ is a constituent of a tensor product of simple objects $S_{Y, \pi} \otimes S_{Z, \gamma}$ if and only if there exist elements $t, s \in G$ such that*

- (a) $\text{Hom}_{\mathcal{C}}(U, \rho^t(Y) \otimes \rho^s(Z)) \neq 0$ and
- (b) $m_T(\delta|_T, {}^t\pi|_T \otimes^s \gamma|_T \otimes \tau_{[U]}^{[Y], [Z]}(t, s)) \neq 0$, where $T = G_U \cap G_{t^s(Y)} \cap G_{t^s(Z)}$.

3. FUSION SUBCATEGORIES OF AN EQUIVARIANTIZATION

In this subsection we shortly recall from [7] the description of fusion subcategories of an equivariantization. Recall that for the rest of the paper we have fixed a system $C = \{c_Y\}_{Y \in \text{Irr}(\mathcal{C})}$ of chosen isomorphisms for any $Y \in \text{Irr}(\mathcal{C})$ satisfying Equation (1.9) and with $c_1^g = \text{id}_1$ for all $g \in G$. Moreover we have $c_Y^1 = \text{id}_Y$ for any simple object Y and the values of $\alpha_Y(g, g')$ are roots of unity for any $g, g' \in G$. By the object $S_{Y, \pi}$ we will understand from now on the simple object $S_{[Y], \pi}^{c_Y}$ associated to the chosen set C .

Let $\pi : G \rightarrow \text{GL}(V)$ be a finite dimensional representation of G on the vector space V . Then the (trivial) object $V \otimes \mathbf{1} \in \mathcal{C}$ has a G -equivariant structure defined by $\pi(g) \otimes \text{id}_1 : {}^t g(V \otimes \mathbf{1}) \rightarrow V \otimes \mathbf{1}$. To see that this defines an equivariant structure one needs to use the fact that $c_1^g = \text{id}_1$. This induces an embedding of fusion categories $\text{Rep } G \rightarrow \mathcal{C}^G$ that gives rise to an exact sequence of fusion categories

$$(3.1) \quad \text{Rep } G \rightarrow \mathcal{C}^G \rightarrow \mathcal{C}.$$

See [3, Subsection 5.4]. Note that if π is an irreducible representation of $G = G_1$ on V , then the simple object $(V \otimes \mathbf{1}, (\pi(g) \otimes \text{id}_1)_g)$ of \mathcal{C}^G is isomorphic to the simple object $S_{1, \pi}$ corresponding to the pair $(\mathbf{1}, \pi)$.

3.1. The fusion datum associated to a fusion subcategory. Let \mathcal{D} be a fusion subcategory of \mathcal{C}^G . One defines the following datum associated to \mathcal{D} . Let $\mathcal{S}_{\mathcal{D}}$ be the fusion subcategory of \mathcal{C} generated by $F(\mathcal{D})$ where F is the forgetful functor $F : \mathcal{C}^G \rightarrow \mathcal{C}$. Thus $\text{Irr}(\mathcal{S}_{\mathcal{D}})$ consists of the set of all $Y \in \text{Irr}(\mathcal{C})$ such that there is some $(X, \mu) \in \mathcal{C}^G$ with Y a subobject of $F(X, \mu)$. Clearly $\mathcal{S}_{\mathcal{D}}$ is a fusion subcategory of \mathcal{C} stable under the action of G . Indeed if Y is a subobject of $F(S)$ with S a simple object of \mathcal{C}^G then ${}^t g(Y)$ is a subobject of $F({}^t g(S)) \cong F(S)$.

Define also a normal subgroup $H_{\mathcal{D}}$ of G by the equality $\mathcal{D} \cap \text{Rep}(G) = \text{Rep}(G/H_{\mathcal{D}})$. Note that the normal subgroup $H_{\mathcal{D}}$ satisfies also the following equation:

$$(3.2) \quad H_{\mathcal{D}} = \bigcap_{\{\gamma \mid S_{1, \gamma} \in \mathcal{D}\}} \ker_G \gamma.$$

3.2. The dual of a simple object. Let $Y \in \text{Irr}(\mathcal{C})$. Then the multiplicity of the unit object of \mathcal{C} in the tensor product $Y \otimes Y^*$ is one. Hence $\tau_Y := \tau_1^{Y, Y^*} = \text{Hom}_{\mathcal{C}}(\mathbf{1}, Y \otimes Y^*)$ is a one dimensional (linear) representation of $G = G_1$. In particular, it follows that the cohomology class of the product $\alpha_Y \alpha_{Y^*}$ is trivial on the subgroup $G_Y = G_{Y^*}$. Recall that the dual π^* of the G_Y -projective representation π is defined as V_π^* with $\pi^*(h)(f) = f \circ \pi(h)^{-1}$. This is an α_Y^{-1} -projective representation of G_Y . Moreover in these conditions [6, Proposition 3.12] implies that

$$(3.3) \quad S_{Y, \pi}^* = S_{Y^*, \pi^* \tau_Y^{-1}}.$$

Lemma 3.1. *One has that $S_{1, \delta}$ is a constituent of $S_{Y, \pi} \otimes (S_{Y, \gamma})^*$ if and only if δ is a constituent of $(\pi \gamma^*) \uparrow_{G_Y}^G$.*

Proof. Applying Corollary 2.6 for $r = s = 1$ it follows $S_{1, \delta}$ is a constituent of $S_{Y, \pi} \otimes (S_{Y, \gamma})^*$ if and only if δ is a constituent of $\pi \gamma^* \tau_Y^{-1} \tau_Y = \pi \gamma^*$. The statement of the lemma follows now by Frobenius reciprocity. \square

Lemma 3.2. *Suppose that \mathcal{D} is a fusion subcategory of \mathcal{C}^G . Then $H_{\mathcal{D}} \subset \text{core}_G(G_Y)$ for all $Y \in \text{Irr}(\mathcal{S}_{\mathcal{D}})$. If $S_{Y, \gamma} \in \mathcal{D}$ and $S_{Y, \gamma'} \in \mathcal{D}$ then for all $h \in H$ one has*

$$\frac{\chi_\gamma(h)}{\chi_\gamma(1)} = \frac{\chi_{\gamma'}(h)}{\chi_{\gamma'}(1)} = \lambda_{\mathcal{D}}(Y, h)$$

for some root of unity $\lambda_{\mathcal{D}}(Y, h) \in k^*$.

Proof. If $S_{Y, \gamma} \in \mathcal{D}$ and $S_{Y, \gamma'} \in \mathcal{D}$ then $S_{Y, \gamma} \otimes (S_{Y, \gamma'})^* \in \mathcal{D}$. But $(S_{Y, \gamma'})^* = S_{Y^*, \gamma'^* \tau_Y^{-1}}$ by Equation (3.3). Then from Lemma 3.1 it follows that for all δ , a constituent of $(\gamma \gamma'^* \tau_Y \tau_Y^{-1}) \uparrow_{G_Y}^G = (\gamma \gamma'^*) \uparrow_{G_Y}^G$, one has that $S_{1, \delta} \in \mathcal{D}$. Thus $H_{\mathcal{D}}$ acts as a scalar on δ and therefore on $(\gamma \gamma'^*) \uparrow_{G_Y}^G$. By Equation (3.2) we have $H_{\mathcal{D}} \subset \ker_G((\gamma \gamma'^*) \uparrow_{G_Y}^G)$. Then Lemma 7.4 implies the first statement.

The second statement follows as well from this lemma since for all $h \in H_{\mathcal{D}}$ one has $|\chi_\gamma(h)| \leq \chi_\gamma(1)$ and $\chi_{\gamma'^*}(h) = \overline{\chi_{\gamma'}(h)}$. Thus if $(\gamma \gamma'^*)(h) = (\gamma \gamma'^*)(1)$ then by Lemma 7.2 there is a root of unity $\lambda_{\mathcal{D}}(Y, h)$ such that

$$\frac{\chi_\gamma(h)}{\chi_\gamma(1)} = \frac{\chi_{\gamma'}(h)}{\chi_{\gamma'}(1)} = \lambda_{\mathcal{D}}(Y, h).$$

\square

For any fusion subcategory $\mathcal{D} \subset \mathcal{C}^G$ define a function $\lambda_{\mathcal{D}} : \text{Irr}(\mathcal{S}_{\mathcal{D}}) \times H \rightarrow k^*$ as following:

$$(3.4) \quad \lambda_{\mathcal{D}}([Y], h) = \frac{\chi_\pi(h)}{\chi_\pi(1)}$$

if $S_{Y,\pi} \in \mathcal{D}$. Note that $\lambda_{\mathcal{D}}$ is well defined since if $Y \simeq Y'$ then $S_{Y',\pi} \simeq S_{Y,\pi} \in \mathcal{D}$ by Remark 2.2.

Lemma 3.3. *Let $Y \in \mathcal{S}_{\mathcal{D}}$ be a simple object. With the above notations note that $S_{Y,\delta} \in \mathcal{D}$ if and only if $\frac{\chi_{\delta}(h)}{\chi_{\delta}(1)} = \lambda_{\mathcal{D}}([Y], h)$ for all $h \in H_{\mathcal{D}}$.*

Proof. If $S_{Y,\delta} \in \mathcal{D}$ then clearly $\frac{\chi_{\delta}(h)}{\chi_{\delta}(1)} = \lambda_{\mathcal{D}}([Y], h)$ by definition.

Conversely, since $Y \in \mathcal{S}_{\mathcal{D}}$ then there is a α_Y -projective representation π of G_Y such that $S_{Y,\pi} \in \mathcal{D}$. Thus $\frac{\chi_{\pi}(h)}{\chi_{\pi}(1)} = \lambda_{\mathcal{D}}([Y], h)$ for all $h \in H_{\mathcal{D}}$.

Note that as above one has that $S_{1,(\gamma\delta^*)\uparrow_{G_Y}^G}$ is a constituent of $S_{Y,\gamma} \otimes S_{Y,\delta}^*$. But if $\frac{\chi_{\delta}(h)}{\chi_{\delta}(1)} = \lambda_{\mathcal{D}}([Y], h)$ then $H_{\mathcal{D}}$ acts as a identity on the representation $\gamma\delta^*$. Therefore by Lemma 7.4 $H_{\mathcal{D}}$ acts as a identity also on $(\gamma\delta^*)\uparrow_{G_Y}^G$. Thus $S_{1,(\gamma\delta^*)\uparrow_{G_Y}^G} \in \mathcal{D}$ and therefore $S_{Y,\delta} \in \mathcal{D}$ since

$$\text{Hom}_{\mathcal{C}^G}(S_{Y,\delta}, S_{1,(\gamma\delta^*)\uparrow_{G_Y}^G}^* \otimes S_{Y,\pi}) \simeq \text{Hom}_{\mathcal{C}^G}(S_{1,(\gamma\delta^*)\uparrow_{G_Y}^G}, S_{Y,\pi} \otimes S_{Y,\delta}^*) \neq 0.$$

□

Lemma 3.4. *Let \mathcal{D} be a fusion subcategory of \mathcal{C}^G . Then $(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}})$ is a fusion datum as in Definition 1.2.*

Proof. By Lemma 2.3 it is clear that $\mathcal{S}_{\mathcal{D}}$ is G -stable. Moreover $H_{\mathcal{D}}$ is a normal subgroup of G and it acts trivially on $\text{Irr}(\mathcal{S}_{\mathcal{D}})$ since $H_{\mathcal{D}} \subset G_Y$ for all $Y \in \text{Irr}(\mathcal{S}_{\mathcal{D}})$. On the other hand since H acts as a scalar on each projective representation π with $S_{Y,\pi} \in \mathcal{D}$ it follows that $\alpha_Y|_H$ is cohomologous trivial for any simple object Y of $\mathcal{S}_{\mathcal{D}}$. This proves the first three conditions from the definition of a fusion datum.

Suppose that $S_{Y,\gamma} \in \mathcal{D}$ and let $h, l \in H_{\mathcal{D}}$. Since by Lemma 3.2 $k_{\alpha_{[Y]}|_{H_{\mathcal{D}}}} H_{\mathcal{D}}$ acts as a scalar on γ it follows that

$$\left(\frac{\chi_{\gamma}(hl)}{\chi_{\gamma}(1)}\right) = \alpha_{[Y]}(h, l) \left(\frac{\chi_{\gamma}(h)}{\chi_{\gamma}(1)}\right) \left(\frac{\chi_{\gamma}(l)}{\chi_{\gamma}(1)}\right).$$

Then

$$\begin{aligned} \lambda_{\mathcal{D}}([Y], hl) &= \frac{\chi_{\gamma}(hl)}{\chi_{\gamma}(1)} = \frac{\chi_{\gamma}(h)}{\chi_{\gamma}(1)} \frac{\chi_{\gamma}(l)}{\chi_{\gamma}(1)} \alpha_{[Y]}(h, l) \\ &= \lambda_{\mathcal{D}}([Y], h) \lambda_{\mathcal{D}}([Y], l) \alpha_{[Y]}(h, l). \end{aligned}$$

for any $Y \in \text{Irr}(\mathcal{S}_{\mathcal{D}})$ and any $h, l \in H_{\mathcal{D}}$. This proves Equation (1.13).

Now suppose that $U \in \text{Irr}(\mathcal{S}_{\mathcal{D}})$ such that $\text{Hom}_{\mathcal{C}}(U, \mathfrak{t}^t(Y) \otimes \mathfrak{t}^s(Z)) \neq 0$. Then from the definition of $\mathcal{S}_{\mathcal{D}}$ there are γ, γ', δ irreducible projective representations of respectively G_Y, G_Z, G_U such that $S_{Y,\gamma}, S_{Z,\gamma'} \in \mathcal{D}$ and $\text{Hom}_{\mathcal{C}^G}(S_{U,\delta}, S_{Y,\gamma} \otimes S_{Z,\gamma'}) \neq 0$. For example, by [6, Lemma 2.15]

one can take $\delta := \text{Hom}_{\mathcal{C}}(U, S_{Y,\gamma} \otimes S_{Z,\gamma'})$. From Corollary 2.6 this implies that δ is a constituent of $({}^t\gamma \otimes {}^s\gamma' \otimes \tau_U^{Y,Z}(s, t)) \uparrow_T^{G_U}$. Since by Lemma 3.2 $H_{\mathcal{D}}$ acts by the same scalar on each such constituent δ it follows that $H_{\mathcal{D}}$ acts as a scalar on the whole induced representation $({}^t\gamma \otimes {}^s\gamma' \otimes \tau_U^{Y,Z}(s, t)) \uparrow_T^{G_U}$. On the other hand by Lemma 7.3 it follows that $H_{\mathcal{D}}$ acts as a scalar on ${}^t\pi \otimes {}^s\delta \otimes \tau_U^{Y,Z}(s, t)$. Since $H_{\mathcal{D}}$ acts also as a scalar on ${}^t\pi$ and ${}^s\delta$ it follows that $H_{\mathcal{D}}$ acts as a scalar on $\tau_U^{Y,Z}(t, s)$ which proves the fourth condition of a fusion datum.

Then one has that

$$\frac{\chi_{\delta}(h)}{\chi_{\delta}(1)} = \frac{\chi_{(\gamma|_{G_U \cap G_Y \cap G_Z} \otimes \gamma'|_T)^{G_Z} \otimes \tau_U^{Y,Z}}(h)}{\chi_{(\gamma|_{G_U \cap G_Y \cap G_Z} \otimes \gamma'|_T)^{G_Z} \otimes \tau_U^{Y,Z}}(1)} = \frac{\chi_{\gamma}(h)}{\chi_{\gamma}(1)} \frac{\chi_{\gamma'}(h)}{\chi_{\gamma'}(1)} \frac{\chi_{\tau_U^{Y,Z}}(h)}{\chi_{\tau_U^{Y,Z}}(1)}$$

for all $h \in H_{\mathcal{D}}$. This shows that

$$\begin{aligned} \lambda_{\mathcal{D}}([Y], h) \lambda_{\mathcal{D}}([Z], h) &= \left(\frac{\chi_{\gamma}(h)}{\chi_{\gamma}(1)} \frac{\chi_{\gamma'}(h)}{\chi_{\gamma'}(1)} \right) = \frac{\chi_{\delta}(h)}{\chi_{\delta}(1)} \left(\frac{\chi_{\tau_U^{Y,Z}}(h)}{\chi_{\tau_U^{Y,Z}}(1)} \right)^{-1} \\ &= \lambda_{\mathcal{D}}([U], h) \left(\frac{\chi_{\tau_U^{Y,Z}}(h)}{\chi_{\tau_U^{Y,Z}}(1)} \right)^{-1} \end{aligned}$$

which proves Equation (1.14).

As noticed in Lemma 2.3 for any $g \in G$ one has that

$$S_{Y,\pi} \cong S_{t^g(Y), {}^gD_{g,[Y]}(-) {}^g\pi},$$

Therefore if $S_{Y,\pi} \cong S_{t^g(Y), {}^gD_{g,[Y]}(-) {}^g\pi} \in \mathcal{D}$ then one has that

$$\begin{aligned} \lambda_{\mathcal{D}}([t^g(Y)], h) &= \frac{{}^gD_{g,[Y]}(h) {}^g\pi(h)}{{}^gD_{g,[Y]}(1) {}^g\pi(1)} = \frac{D_{g,[Y]}(g^{-1}hg) \chi_{\pi}(g^{-1}hg)}{\chi_{\pi}(1)} \\ &= \lambda_{\mathcal{D}}([Y], g^{-1}hg) D_{g,[Y]}(g^{-1}hg) \end{aligned}$$

which proves Equation (1.15). \square

3.3. Construction of a fusion subcategory from a fusion datum.

Conversely suppose we have a fusion datum $(\mathcal{S}, H, \lambda)$ as in Definition 1.2.

To this datum we associate the fusion subcategory $\mathcal{C}(\mathcal{S}, H, \lambda)$ generated by all the simple objects $S_{Y,\pi} \in \text{Irr}(\mathcal{C})$ that satisfies

$$(3.5) \quad \frac{\chi_{\pi}(h)}{\chi_{\pi}(1)} = \lambda([Y], h)$$

for all $h \in H$.

Theorem 3.5. $\mathcal{C}(\mathcal{S}, H, \lambda)$ as defined above is a fusion subcategory of \mathcal{C}^G .

Proof. It is enough to show only that $\mathcal{C}(\mathcal{S}, H, \lambda)$ is closed under tensor products. Suppose that two simple objects $S_{Y,\pi}, S_{Z,\gamma} \in \mathcal{C}(\mathcal{S}, H, \lambda)$. We have to show that all the simple constituents $S_{U,\delta}$ of the tensor product $S_{Y,\pi} \otimes S_{Z,\gamma}$ are also in $\mathcal{C}(\mathcal{S}, H, \lambda)$. Using Corollary 2.6 it follows that there are t, s such that U is a constituent of $t^t(Y) \otimes t^s(Z)$ and $\text{Hom}_T(\delta|_T, {}^t\pi|_T \otimes {}^s\gamma|_T \otimes \tau_U^{Y,Z}(t, s)) \neq 0$. Since H acts as a scalar on the representation ${}^t\pi|_T \otimes {}^s\gamma|_T \otimes \tau_U^{Y,Z}(s, t)$ it follows that

$$(3.6) \quad \frac{\chi_\delta(h)}{\chi_\delta(1)} = \frac{\chi_{{}^t\pi}(h)}{\chi_{{}^t\pi}(1)} \frac{\chi_{{}^s\gamma}(h)}{\chi_{{}^s\gamma}(1)} \frac{\chi_{\tau_U^{Y,Z}(s,t)}(h)}{\chi_{\tau_U^{Y,Z}(s,t)}(1)} = \frac{\chi_\pi(t^{-1}ht)}{\chi_\pi(1)} \frac{\chi_\gamma(s^{-1}hs)}{\chi_\gamma(1)} \frac{\chi_{\tau_U^{Y,Z}(t,s)}(h)}{\chi_{\tau_U^{Y,Z}(s,t)}(1)}$$

Note that in this case by applying Equation (2.8) one has:

$$\begin{aligned} \lambda([U], h) &= \lambda([t^t(Y)], h) \lambda([t^s(Z)], h) \frac{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(h)}{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(1)} \\ &= \lambda([Y], t^{-1}ht) D_{t,[Y]}(t^{-1}ht) \lambda([Z], t^{-1}ht) D_{s,[Z]}(s^{-1}hs) \frac{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(h)}{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(1)} \\ &= \lambda([Y], t^{-1}ht) \lambda([Z], t^{-1}ht) [D_{t,[Y]}(t^{-1}ht) D_{s,[Z]}(s^{-1}hs) \frac{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(h)}{\chi_{\tau_{[U]}^{[t^t(Y)], [t^s(Z)]}}(1)}] \\ &= \frac{\chi_{{}^t\pi}(h)}{\chi_{{}^t\pi}(1)} \frac{\chi_{{}^s\gamma}(h)}{\chi_{{}^s\gamma}(1)} \frac{\chi_{\tau_{[U]}^{[(Y)], [(Z)]}(t,s)}(h)}{\chi_{\tau_{[U]}^{[(Y)], [(Z)]}(t,s)}(1)} \\ &= \frac{\chi_\delta(h)}{\chi_\delta(1)} \end{aligned}$$

We also used Equation (1.14), Equation (1.15) and Proposition 2.5 in the last equality. Thus $\lambda([U], h) = \frac{\delta(h)}{\delta(1)}$ which implies that $S_{U,\delta} \in \mathcal{C}(\mathcal{S}, H, \lambda)$. \square

3.4. Proof of Theorem 1.3. The proof of Theorem 1.3 follows from the following Propositions.

Proposition 3.6. Let $(\mathcal{S}, H, \lambda)$ be a fusion datum and let $\mathcal{C}(\mathcal{S}, H, \lambda) \subseteq \mathcal{C}^G$ be the associated fusion subcategory associated as above. Then the fusion datum $(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}})$ corresponding to $\mathcal{D} := \mathcal{C}(\mathcal{S}, H, \lambda)$ coincides with $(\mathcal{S}, H, \lambda)$.

Proof. Straightforward. \square

Proposition 3.7. *Let $\mathcal{D} \subseteq \mathcal{C}$ be a fusion subcategory and let $(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}})$ be the fusion datum associated to \mathcal{D} . Then $\mathcal{C}(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}}) = \mathcal{D}$*

Proof. It is clear from Remark 3.3 that $\mathcal{C}(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}}) \subseteq \mathcal{D}$. On the other hand the opposite inclusion follows from definitions of $\lambda_{\mathcal{D}}$ and of the fusion datum $(\mathcal{S}_{\mathcal{D}}, H_{\mathcal{D}}, \lambda_{\mathcal{D}})$. \square

Corollary 3.8. *With the above notations one has that $\text{FPdim } \mathcal{C}(\mathcal{S}, H, \lambda) = \frac{|G| \text{FPdim } \mathcal{S}}{|H|}$.*

Proof. One has that

$$\begin{aligned} \text{FPdim } \mathcal{C}(\mathcal{S}, H, \lambda) &= \sum_{S_{Y,\pi} \in \mathcal{C}(\mathcal{S}, H, \lambda)} \text{FPdim}^2 S_{Y,\pi} = \\ &= \sum_{S_{Y,\pi} \in \mathcal{C}(\mathcal{S}, H, \lambda)} \frac{|G|^2}{|G_Y|^2} \text{FPdim}^2 Y \dim^2 \pi \\ &= \sum_{Y \in \text{Irr}(\mathcal{S})/G} \left(\frac{|G|^2}{|G_Y|^2} \text{FPdim}^2 Y \right) \left(\sum_{\pi \in \text{Irr}_{\alpha_Y}(G_Y), \pi|_H = \lambda(Y, -)} \dim^2 \pi \right) \end{aligned}$$

Then [16, Lemma 5.8] shows that

$$\sum_{\pi \in \text{Irr}_{\alpha_Y}(G_Y), \pi|_H = \lambda(Y, -)} \dim^2 \pi = \frac{|G_Y|}{|H|}$$

and the equality $\text{FPdim } \mathcal{C}(\mathcal{S}, H, \lambda) = \frac{|G| \text{FPdim } \mathcal{S}}{|H|}$ follows from the equality $\text{FPdim } \mathcal{S} = \sum_{Y \in \text{Irr}(\mathcal{S})/G} \frac{|G|}{|G_Y|} \text{FPdim}^2 Y$. \square

Remark 3.9. Note that if $H = 1$ then necessarily $\lambda : \text{Irr}(\mathcal{D}) \rightarrow k^*$ is trivial since $\lambda([Y], 1) = 1$ for all $Y \in \text{Irr}(\mathcal{S})$.

Proposition 3.10. *Any fusion subcategory of \mathcal{C}^G is also an equivariantization of a fusion subcategory of \mathcal{C} by a quotient subgroup of G .*

Proof. One has the exact sequence

$$(3.7) \quad \text{Rep } G \rightarrow \mathcal{C}^G \rightarrow \mathcal{C}$$

and its subsequence

$$(3.8) \quad \text{Rep}(G/H) \rightarrow \mathcal{C}(H, \mathcal{S}, \lambda) \rightarrow \mathcal{S}$$

which is *central* since the previous sequence is central. It follows by [4, Theorem 3.6] that the second sequence is also an equivariantization, i.e. $\mathcal{C}(H, \mathcal{S}, \lambda) = \mathcal{S}^{G/H}$ for some action of G/H on \mathcal{S} . \square

3.5. On the lattice of fusion subcategories.

Proposition 3.11. *With the above notations one has that $\mathcal{C}(\mathcal{S}, H, \lambda) \subset \mathcal{C}(\mathcal{S}', H', \lambda')$ if and only if $\mathcal{S} \subset \mathcal{S}'$, $H \supset H'$ and $\lambda|_{H' \times \text{Irr}(\mathcal{S})} = \lambda'|_{H' \times \text{Irr}(\mathcal{S})}$.*

Proof. If $\mathcal{C}(\mathcal{S}, H, \lambda) \subset \mathcal{C}(\mathcal{S}', H', \lambda')$ then clearly

$$\mathcal{S} = F(\mathcal{C}(\mathcal{S}, H, \lambda)) \subset F(\mathcal{C}(\mathcal{S}', H', \lambda')) = \mathcal{S}'.$$

Moreover $\text{Rep}(G/H) = \text{Rep}(G) \cap \mathcal{C}(\mathcal{S}, H, \lambda) \subset \text{Rep}(G) \cap \mathcal{C}(\mathcal{S}', H', \lambda') = \text{Rep}(G/H')$ which implies $H \supset H'$. Suppose that $S_{X,\pi} \in \mathcal{C}(\mathcal{S}, H, \lambda)$. It follows that $\alpha_X|_{H'} = 1$. Then definition of λ and λ' implies that

$$(3.9) \quad \lambda([X], h) = \frac{\chi_{\pi(h)}}{\chi_{\pi(1)}} = \lambda'([X], h)$$

Thus the other equality $\lambda|_{H' \times \text{Irr}(\mathcal{S})} = \lambda'|_{H' \times \text{Irr}(\mathcal{S})}$ also holds.

The converse follows directly from the definition of $\mathcal{C}(\mathcal{S}, H, \lambda)$. \square

Suppose there are given two fusion datum $(\mathcal{S}, H, \lambda)$ and $(\mathcal{S}', H', \lambda')$. Define

$$(3.10) \quad \phi_{\lambda, \lambda'} : \text{Irr}(\mathcal{S} \cap \mathcal{S}') \times H \cap H' \rightarrow k^*$$

by $\phi_{\lambda, \lambda'}([Y], h) = \lambda([Y], h)\lambda'^{-1}([Y], h)$ for all $h \in H$.

Let also $\ker \phi_{\lambda, \lambda'}$ be the abelian subcategory of $\mathcal{S} \cap \mathcal{S}'$ consisting of those objects $[Y] \in \text{Irr}(\mathcal{S} \cap \mathcal{S}')$ such that $\phi_{\lambda, \lambda'}([X], h) = 1$ for all $h \in H \cap H'$. Equation (1.14) implies that $\ker \phi_{\lambda, \lambda'}$ is a fusion subcategory of \mathcal{C} . Define also

$$(3.11) \quad \psi_{\lambda, \lambda'} : \text{Irr}(\ker \phi_{\lambda, \lambda'}) \times HH' \rightarrow k^*$$

defined on generators by

$$\psi_{\lambda, \lambda'}([X], hh') = \lambda([X], h)\lambda'([X], h')$$

It is not difficult to see that $\psi_{\lambda, \lambda'}$ is well defined. Indeed, suppose that $hh' = ll'$ with $h, l \in H$ and $h', l' \in H'$. Then $l^{-1}h = l'h'^{-1} \in H \cap H'$. Thus for any $X \in \text{Irr} \ker \phi_{\lambda, \lambda'}$ one has that:

$$\lambda([X], l)^{-1}\lambda([X], h) = \lambda([X], l^{-1}h) = \lambda([X], l'h'^{-1}) = \lambda([X], l')\lambda([X], h')^{-1}$$

which shows that

$$\lambda([X], h)\lambda([X], h') = \lambda([X], l)\lambda([X], l').$$

i.e. $\psi_{\lambda, \lambda'}([X], -)$ is well defined.

Proposition 3.12. *With the above notations one has that $\mathcal{C}(H, \mathcal{S}, \lambda) \cap \mathcal{C}(H', \mathcal{S}', \lambda') = \mathcal{C}(\ker \phi_{\lambda, \lambda'}, HH', \psi_{\lambda, \lambda'})$.*

Proof. Suppose that $\mathcal{C}(\mathcal{S}, H, \lambda) \cap \mathcal{C}(\mathcal{S}', H', \lambda') = \mathcal{C}(\mathcal{S}'', H'', \lambda'')$. If $S_{X,\pi} \in \mathcal{C}(\mathcal{S}, H, \lambda) \cap \mathcal{C}(\mathcal{S}', H', \lambda')$ then one has that $\lambda([X], h) = \frac{\chi_\pi(h)}{\chi_\pi(1)} = \lambda'([X], h)$ which shows that $X \in \text{Irr}(\ker \phi_{\lambda,\lambda'})$. Thus $\mathcal{S}'' \subseteq \ker \phi_{\lambda,\lambda'}$.

Moreover note that $\text{Rep}(G) \cap (\mathcal{C}(\mathcal{S}, H, \lambda) \cap \mathcal{C}(\mathcal{S}', H', \lambda')) = \text{Rep}(G/H) \cap \text{Rep}(G/H') = \text{Rep}(G/HH')$ which implies that $H'' = HH'$. Then by its definition one has $\lambda''([X], hh') = \frac{\chi_\pi(hh')}{\chi_\pi(1)}$. Using Lemma 3.2 it follows that $\lambda''([X], hh') = \alpha_{[X]}(h, h') \frac{\chi_\pi(h)}{\chi_\pi(1)} \frac{\chi_\pi(h')}{\chi_\pi(1)} = \alpha_{[X]}(h, h') \lambda([X], h) \lambda'([X], h') = \psi_{\lambda,\lambda'}([X], h)$ which also shows that $\psi_{\lambda,\lambda'}$ is well defined. \square

Proposition 3.13. *With the above notations one has that $\mathcal{C}(\mathcal{S}, H, \lambda) \vee \mathcal{C}(\mathcal{S}', H', \lambda') = \mathcal{C}(\mathcal{S} \vee \mathcal{S}', H_\vee, \lambda_\vee)$ where*

$$H_\vee = \bigcap_{X \in \text{Irr}(\mathcal{S} \cap \mathcal{S}')} \ker_G[\lambda([X], -) \uparrow_H^G \overline{\lambda'([X^*], -)} \uparrow_{H'}^G]$$

and

$$\lambda_\vee([U], h) := \lambda([Y], h) \lambda'([Z], h)$$

for all $Y \in \text{Irr}(\mathcal{S})$, $Z \in \text{Irr}(\mathcal{S}')$ and all simple objects U with $\text{Hom}_{\mathcal{C}}(U, Y \otimes Z) \neq 0$.

Proof. Clearly $F(\mathcal{C}(\mathcal{S}, H, \lambda) \vee \mathcal{C}(\mathcal{S}', H', \lambda')) = \mathcal{S} \vee \mathcal{S}'$. Moreover $\text{Rep}(G/H \cap H') = \text{Rep}(G/H) \vee \text{Rep}(G/H') \subset \mathcal{C}(\mathcal{S}, H, \lambda) \vee \mathcal{C}(\mathcal{S}', H', \lambda')$. On the other hand if $S_{1,M} \in \mathcal{C}_1 \vee \mathcal{C}_2$ then $S_{1,M}$ appears as constituent of $S_{X,\pi} \otimes S_{X^*,\delta}$. It follows that M is a constituent of $(\pi\delta^*) \uparrow_{G_X}^G$. This implies that

$$H_\vee = \bigcap_{X \in \mathcal{S}_1 \cap \mathcal{S}_2} \ker_G(\lambda([X], -) \uparrow_H^G \overline{\lambda'([X^*], -)} \uparrow_{H'}^G).$$

Formula for λ_\vee follows from definition. \square

Remark 3.14. Note that in the previous Proposition one has $H_\vee \subseteq H \cap H'$.

Corollary 3.15. *Suppose that \mathcal{C} is a braided G -crossed fusion category with $K_0(\mathcal{C})$ commutative. Then $H_\vee = H_1 \cap H_2$.*

Proof. It follows from the fact that in a braided fusion category one has that $\text{FPdim } \mathcal{C}_1 \vee \mathcal{C}_2 = \frac{\text{FPdim } \mathcal{C}_1 \text{ FPdim } \mathcal{C}_2}{\text{FPdim } \mathcal{C}_1 \cap \mathcal{C}_2}$, see [8]. \square

4. EQUIVARIANTIZATION OF BRAIDED G -CROSSED FUSION CATEGORIES

In this section we prove Theorem 1.4. It entails a description for Müger's centralizer of all fusion subcategories of a group theoretical braided fusion category.

4.1. Modular fusion categories. Recall that a *braided tensor category* \mathcal{C} is a tensor category equipped for any two objects $X, Y \in \mathcal{C}$ with natural isomorphisms $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ satisfying the two braiding hexagon axioms, see for example [1, 2].

By [2, Proposition 2.1] it follows that for any object $A \in \mathcal{C}$ one has that $c_{A,I} = l_A r_A^{-1}$ and $c_{I,A} = l_A^{-1} r_A$ where $l_A : A \otimes I \rightarrow A$ and $r_A : I \otimes A \rightarrow A$ are the canonical isomorphisms associated to the unit I of \mathcal{C} .

A *twist* on a braided fusion category \mathcal{C} is a natural automorphism $\theta : \text{id}_{\mathcal{C}} \rightarrow \text{id}_{\mathcal{C}}$ satisfying $\theta_1 = \text{id}_1$ and

$$(4.1) \quad \theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) c_{YX} c_{XY}.$$

A braided fusion category is called *premodular* or *ribbon* if it has a twist satisfying $\theta_{X^*} = \theta_X^*$ for all $X \in \mathcal{C}$, see [5]. By [8] a premodular category is a braided fusion category endowed with a spherical structure. Recall that the entries of the S -matrix, $S = \{s_{X,Y}\}$ of a premodular category are defined as the quantum trace $s_{X,Y} := \text{tr}_q(c_{YX} c_{XY})$, see [1]. A premodular category \mathcal{C} is called *modular* if the above S -matrix is nondegenerate, see [22].

Centralizers in braided fusion categories. Let \mathcal{C} be a braided fusion category and \mathcal{D} be a fusion subcategory of \mathcal{C} . Müger introduced the notion of *centralizer of \mathcal{D}* as the fusion subcategory \mathcal{D}' of \mathcal{C} generated by all simple objects X of \mathcal{C} satisfying

$$(4.2) \quad c_{X,Y} c_{Y,X} = \text{id}_{X \otimes Y}$$

for all objects $Y \in \mathcal{D}$.

In the case of a ribbon category \mathcal{C} the condition of Equation (4.2) is equivalent to

$$(4.3) \quad s_{X,Y} = \text{FPdim}(X) \text{FPdim}(Y)$$

for all objects $Y \in \mathcal{D}$. Note that in general

$$(4.4) \quad |s_{X,Y}| \leq \text{FPdim}(X) \text{FPdim}(Y)$$

by [13, Proposition 2.5]. In the situation of equality in Equation (4.2) we say that the objects X and Y centralize each other.

We also say that X, Y *projectively centralize* each other if and only if there is $\omega \in k^*$ a root of unity such that $s_{X,Y} = \omega \text{FPdim}(X) \text{FPdim}(Y)$.

Let \mathcal{C} be a braided fusion category. We say that a Tannakian subcategory $\mathcal{E} \subset \mathcal{D}$ is *Lagrangian* if $\mathcal{E}' = \mathcal{E}$.

Recall that the *Müger center* of a braided fusion category is the Müger centralizer $\mathcal{Z}_2(\mathcal{C}) = \mathcal{C}'$ of the whole category \mathcal{C} . A braided fusion category is called *nondegenerate* if its Müger center is trivial. If \mathcal{C} is a braided fusion category then by [8, Theorem 3.23, Theorem 3.24] one has that

$$(4.5) \quad \text{FPdim}(\mathcal{D})\text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{D} \cap \mathcal{Z}_2(\mathcal{C})).$$

Moreover if \mathcal{D} is nondegenerate then there is an equivalence of braided fusion categories $\mathcal{C} \simeq \mathcal{D} \boxtimes \mathcal{D}'$. In this case \mathcal{C} is nondegenerate if and only if \mathcal{D}' is. On the other hand by [8, Corollary 3.11] one has that $\mathcal{D}'' = \mathcal{D} \vee \mathcal{Z}_2(\mathcal{C})$. In particular if \mathcal{C} is nondegenerate then $\mathcal{D}'' = \mathcal{D}$. Note that for modular fusion categories the above properties of the centralizers were proven in [13, 12].

Let \mathcal{D} and \mathcal{E} be fusion subcategories of a modular fusion category \mathcal{C} . Then one also has

$$(4.6) \quad (\mathcal{D} \vee \mathcal{E})' = \mathcal{D}' \cap \mathcal{E}' \text{ and } (\mathcal{D} \cap \mathcal{E})' = \mathcal{D}' \vee \mathcal{E}',$$

see [13]. A braided fusion category is called *slightly nondegenerate* if its Müger center is braided equivalent to the category SuperVec of super vector spaces.

Recall that a braided fusion category \mathcal{C} is called *symmetric* if $\mathcal{C}' = \mathcal{C}$. Equivalently, $c_{X,Y}c_{Y,X} = \text{id}_{X \otimes Y}$ for all objects $X, Y \in \mathcal{C}$. In this case the braiding c is also called symmetric.

4.2. Braided G -crossed fusion categories. In this section we recall a description (due to Kirillov, Jr. [10] and Müger [11]) of the structure of braided fusion categories containing a subcategory $\text{Rep}(G)$ for some finite group G as a braided fusion subcategory. They are described as equivariantizations of *braided G -crossed categories*, which were introduced by Turaev [23].

Recall that a braided G -crossed fusion category consists of the following data:

- (1) G a finite group,
- (2) \mathcal{C} a fusion category with a (not necessarily faithful) G -grading

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,$$

- (3) an action of G $t : G \rightarrow \underline{\text{Aut}}_{\otimes} \mathcal{C}$ on \mathcal{C} by tensor autoequivalences, satisfying $t_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$.

(4) a family of natural isomorphisms, $c_{X,Y} : X \otimes Y \xrightarrow{\cong} t^g(Y) \otimes X$ for any $X \in \mathcal{C}_g$, $Y \in \mathcal{C}$, called *G-braidings*, satisfying the following compatibility conditions:

$$(4.7) \quad ((t_2^{g,h})_Y^{-1} \otimes \text{id}_{T_g(X)}) \circ ((t_2^{ghg^{-1},g})_Y \otimes \text{id}_{T_g(X)}) \circ c(t^g(X), t^g(Y)) \circ (t_2^g)_{X,Y} = \\ = (t_2^g)_{t^h(Y),X} \circ t^g(c(X,Y)),$$

for all $g, h \in G$ and objects $X \in \mathcal{C}_h$, $Y \in \mathcal{C}$.

$$(4.8) \quad \alpha_{t^g(t^h(Z)),X,Y}^{-1} \circ ((t_{g,h}^2)_Z^{-1} \otimes \text{id}_{X \otimes Y}) \circ c(X \otimes Y, Z) \circ \alpha_{X,Y,Z}^{-1} = \\ = (c(X, t^h(Z)) \otimes \text{id}_Y) \circ \alpha_{X,t^h(Z),Y}^{-1} \circ (\text{id}_X \otimes c(Y, Z)),$$

for all $g, h \in G$ and objects $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $Z \in \mathcal{C}$.

$$(4.9) \quad \alpha_{t^g(Y),t^g(Z),X} \circ ((t_2^g)_{Y,Z} \otimes \text{id}_X) \circ c(X, Y \otimes Z) \circ \alpha_{X,Y,Z} = \\ = (\text{id}_{t^g(Y)} \otimes c(X, Z)) \circ \alpha_{t^g(Y),X,Z} \circ (c(X, Y) \otimes \text{id}_Z),$$

It is well known that in this case $c_{1,X} = \text{id}_X = c_{X,1}$ for any object X of \mathcal{C} .

For any simple object X of \mathcal{C} we define the degree $\deg(X) \in G$ of X , to be the element that satisfies $X \in \mathcal{C}_{\deg(X)}$.

Remark 4.1. Note that the trivial component \mathcal{C}_1 of a braided G -crossed fusion category \mathcal{C} is a braided fusion category with the action of G on it given by *braided autoequivalences*. This can be seen by taking $X, Y \in \mathcal{C}_1$ in the second and the third compatibility condition from above.

It is well known that in this case \mathcal{C}^G is a braided fusion category with the braiding $\tilde{c}_{S,T} : S \otimes T \rightarrow T \otimes S$ given by

$$(4.10) \quad S \otimes T = \bigoplus_{g \in K_S} S_g \otimes T \xrightarrow{\oplus_{g \in K_S} c_{S_g,T}} \bigoplus_{g \in K_S} t^g(T) \otimes S \xrightarrow{\oplus_{g \in K_S} (\mu_T^g \otimes 1)} \bigoplus_{g \in K_S} T \otimes S_g = T \otimes S.$$

for all $S, T \in \mathcal{C}^G$.

5. MÜGER CENTRALIZER FOR THE EQUIVARIANTIZATION OF A BRAIDED G -CROSSED FUSION CATEGORIES

Suppose that $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ is a braided G -crossed fusion category with $\mathcal{D} = \mathcal{C}_1$. Let $G_1 := \{g \in G \mid \mathcal{C}_g \neq 0\}$. Then clearly G_1 is a normal subgroup of G .

For any object $S = \bigoplus_{g \in G} S_g \in \mathcal{C}$ define its support K_S as $K_S := \{g \in G \mid S_g \neq 0\}$. Let \mathcal{C}^G be the equivariantized fusion category and $F : \mathcal{C}^G \rightarrow \mathcal{C}$ be the forgetful functor. By abuse of notations sometimes we identify $F(X)$ with X for any object $X \in \mathcal{C}^G$. For this reason, for

any $S \in \mathcal{C}^G$ we also sometimes denote by K_S the support of the object $F(S) \in \mathcal{C}$.

Remark 5.1. Let $M = \bigoplus_{g \in K_M} M_g$ be an object of \mathcal{C}^G . Since $\mu_T^g : \mathfrak{t}^g(M) \rightarrow M$ is an isomorphism in \mathcal{C} it follows that $\mathfrak{t}^g(M_h) \simeq M_{ghg^{-1}}$ by the morphism $\mu_M^g|_{M_h}$.

Lemma 5.2. *Suppose that \mathcal{C} is a braided G -crossed fusion category as above. Let $S = \bigoplus_{g \in K_S} S_g$ and $T = \bigoplus_{g \in K_T} T_g$ be two objects of \mathcal{C}^G with $S_g, T_g \in \mathcal{C}_g$. Then S and T projectively centralize each other in \mathcal{C}^G if and only if the following conditions are satisfied:*

- (1) K_S commutes elementwise with K_T .
- (2) There is a root of unity $\omega \in k^*$ such that:

$$(5.1) \quad (\mu_S^h|_{\mathfrak{t}^h(S_g)} \otimes \text{id}_{T_h}) c_{T_h, S_g} (\mu_T^g|_{\mathfrak{t}^g(T_h)} \otimes \text{id}_{S_g}) c_{S_g, T_h} = \omega \text{id}_{S_g \otimes T_h}$$

for all $g \in K_S$ and $h \in K_T$.

Proof. From Equation (4.10) the braiding $\tilde{c}_{S,T}$ between S and T can be written on the components as following:

$$(5.2) \quad \tilde{c}_{S,T}^{g,h} : S_g \otimes T_h \xrightarrow{c_{S_g, T_h}} \mathfrak{t}^g(T_h) \otimes S_g \xrightarrow{\mu_T^g|_{\mathfrak{t}^g(T_h)} \otimes \text{id}_{S_g}} T_{ghg^{-1}} \otimes S_g$$

It follows that the braiding $\tilde{c}_{S,T}$ sends the homogenous subobject $S_g \otimes T_h$ of $S \otimes T$ into the homogenous subobject $T_{ghg^{-1}} \otimes S_g$ of $T \otimes S$. Applying now $\tilde{c}_{T,S}$ one has that $\tilde{c}_{T,S}^{ghg^{-1},g}$ is given by

$$(5.3) \quad T_{ghg^{-1}} \otimes S_g \xrightarrow{c_{T_{ghg^{-1}}, S_g}} \mathfrak{t}^{ghg^{-1}}(S_g) \otimes T_{ghg^{-1}} \xrightarrow{\mu_S^{ghg^{-1}}|_{\mathfrak{t}^{ghg^{-1}}(S_g)} \otimes \text{id}_{T_{ghg^{-1}}}} S_{ghg^{-1}g^{-1}} \otimes T_{ghg^{-1}}$$

Thus the homogenous subobject $T_{ghg^{-1}} \otimes S_g$ of $T \otimes S$ is sent by $\tilde{c}_{T,S}$ to $S_{(ghg^{-1})h(ghg^{-1})^{-1}} \otimes T_{ghg^{-1}}$. Since the composition of these two maps should be a scalar multiple of the identity it follows that $ghg^{-1} = h$ i.e g commutes to h . This shows that K_S and K_T commutes elementwise.

Note that imposing that the composition $\tilde{c}_{S,T} \tilde{c}_{T,S}$ of the two braidings being $\omega \text{id}_{S \otimes T}$ it implies the other equation. \square

Remark 5.3. Suppose that $S = \bigoplus_{g \in K_S} S_g$ projectively centralizes $T = \bigoplus_{h \in K_T} T_h$. It follows from previous lemma that $gh = hg$ for any $g \in K_S$ and any $h \in K_T$. On the other hand Remark 5.1 implies that $\mu_T^g|_{\mathfrak{t}^g(T_h)} : \mathfrak{t}^g(T_h) \rightarrow T_h$ is an isomorphism in \mathcal{C} which implies that $g \in G_{T_h}$. Thus $K_S \subset \bigcap_{h \in K_T} G_{T_h}$. Similarly $K_T \subset \bigcap_{g \in K_S} G_{S_g}$ for all $g \in K_S$.

Remark 5.4. The proof of the previous lemma also shows that S and T centralize each other if and only if K_S commutes elementwise with K_T and

$$(5.4) \quad (\mu_S^v|_{\mathfrak{t}^v(S_u)} \otimes \text{id}_{T_v}) c_{T_v, S_u} (\mu_T^u|_{\mathfrak{t}^u(T_v)} \otimes \text{id}_{S_u}) c_{S_u, T_v} = \text{id}_{S_u \otimes T_v}$$

for all $u \in K_S$ and $v \in K_T$.

5.1. Two subgroups associated to a fusion subcategory of \mathcal{C}^G .

For a fusion subcategory $\mathcal{D} \subset \mathcal{C}^G$ define $H_{\mathcal{D}}$, the normal subgroup of G such that $\text{Rep}(G) \cap \mathcal{D} = \text{Rep}(G/H_{\mathcal{D}})$. Let also $K_{\mathcal{D}}$ be the subgroup generated by the support of all simple objects of \mathcal{D} .

Lemma 5.5. *Suppose that \mathcal{C} is a braided G -crossed fusion category as above and let X be an object of \mathcal{C}^G . Then for any $V \in \text{Rep}(G)$ one has that $S_{1,V}$ centralizes X if and only if $K_X \subset \ker_G(V)$.*

Proof. Let $\rho : G \rightarrow \text{End}_k(V)$ be the representation of G associated to V . Recall that $S_{1,V} := V \otimes \mathbf{1}$ is a on object of \mathcal{C}^G with the equivariant structure $t^g(V \otimes \mathbf{1}) \xrightarrow{\rho(g) \otimes 1} V \otimes \mathbf{1}$. On the other hand since $t^1 = \text{id}_{\mathcal{C}}$ note that the braiding

$$\tilde{c}_{S_{1,V}, X} : S_{1,V} \otimes X \xrightarrow{c_{V \otimes \mathbf{1}, X}} t^1(X) \otimes S_{1,V} \rightarrow X \otimes S_{1,V}$$

is the identity morphism on X , via the identification $l_X r_X^{-1}$.

Also, since $t^g(\mathbf{1}) = \mathbf{1}$ the braiding $\tilde{c}_{X, (V \otimes \mathbf{1})} : X \otimes (V \otimes \mathbf{1}) \rightarrow t^g(V \otimes \mathbf{1}) \otimes X = (V \otimes \mathbf{1}) \otimes X$ is given on the homogenous component $X_g \otimes V$ by multiplication with g on V . Thus the double braiding

$$(5.5) \quad c_{S_{1,V}, X} c_{X, S_{1,V}} : X \otimes S_{1,V} \rightarrow S_{1,V} \otimes X$$

regarded as an endomorphism of $X \otimes V$ is given on the component $X_g \otimes V$ just by multiplication by g on V . This implies that the two above objects centralize each other if and only if K_X acts as identity on V , i.e. $K_X \subset \ker_G(V)$. \square

Corollary 5.6. *Suppose that $Y \in \mathcal{C}_g$ is a simple object of \mathcal{C} . Then one has that $S_{1,Y}$ centralizes $S_{Y,\pi}$ if and only if $g \in \ker_G(V)$.*

Proof. Note that $G_Y \subseteq C_G(g)$ and therefore the support of $S_{Y,\pi}$ coincides to K_g , the conjugacy class of G . \square

Corollary 5.7. *One has that $S_{1,V} \in \mathcal{Z}_2(\mathcal{C}^G)$ if and only if $G_1 \subseteq \ker_G(V)$. In particular $S_{1,G/G_1} \in \mathcal{Z}_2(\mathcal{C}^G)$ and $H_{\mathcal{Z}_2(\mathcal{C}^G)} = G_1$.*

Theorem 5.8. *Let \mathcal{C} be a braided G -crossed fusion category and let \mathcal{D} be a fusion subcategory of \mathcal{C}^G . Then with the above notations one has that $K_{\mathcal{D}'}$ and $K_{\mathcal{D}}$ commute elementwise and the following equalities hold $H_{\mathcal{D}'} = K_{\mathcal{D}}$. Moreover $K_{\mathcal{D}'} = H_{\mathcal{D}''} \subseteq H_{\mathcal{D}} \cap G_1$.*

Proof. Lemma 5.2 implies that $K_{\mathcal{D}}$ and $K_{\mathcal{D}'}$ commutes elementwise. On the other hand by Lemma 5.6 one has that $S_{1,V} \in \text{Rep}(G) \cap \mathcal{D}'$ if and only $K_{\mathcal{D}} \subset \ker_G(V)$. This implies that $\text{Rep}(G) \cap \mathcal{D}' = \text{Rep}(G/K_{\mathcal{D}})$ and therefore $H_{\mathcal{D}'} = K_{\mathcal{D}}$. Then $K_{\mathcal{D}'} = H_{\mathcal{D}''} \subseteq H_{\mathcal{D}} \cap G_1$. \square

Proposition 5.9. *Suppose that \mathcal{C} is a braided G -crossed fusion category with a faithful grading by G and \mathcal{C}_1 a nondegenerate braided fusion category. Then for any fusion subcategory $\mathcal{D} \subseteq \mathcal{C}^G$ one has that $H_{\mathcal{D}}$ commutes elementwise to $K_{\mathcal{D}}$ and $H_{\mathcal{D}} = K_{\mathcal{D}'}$.*

Proof. Applying [8, Proposition 4.56] it follows that \mathcal{C}^G is braided nondegenerate. Therefore $\mathcal{D}'' = \mathcal{D}$ and applying the results of the previous theorem for \mathcal{D}' the conclusion follows. \square

Remark 5.10. The above proposition generalizes [16, Proposition 3.6] and [16, Proposition 5.2] to equivariantizations of any braided G -crossed fusion category.

5.2. On the d -morphisms. For any element $g \in G$ denote by K_g the conjugacy class of g in G .

Suppose that are given two simple objects $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ such that K_g commutes elementwise to K_h . For all $a, b \in G$ consider the following isomorphism $d_{X,Y}^{a,b} : t^a(X) \otimes t^b(Y) \rightarrow t^b(Y) \otimes t^a(X)$ in \mathcal{C} given by

$$d_{X,Y}^{a,b} := (t^b(t_2^{aga^{-1},b})_Y (t_2^{b,b^{-1}aga^{-1}b})_Y)^{-1} (c_Y^{b^{-1}aga^{-1}b} \otimes \text{id}_{t^a(X)}) c_{t^a(X),t^b(Y)}$$

Recall that we fixed a set of representatives for the isomorphism classes of simple objects of \mathcal{C} . Suppose that X, Y are simple objects as above such that $X \simeq Y_i$ and $Y \simeq Y_j$. Note that if $u : X \rightarrow Y_i$ and $v : Y \rightarrow Y_j$ are isomorphisms then

$$(5.6) \quad (t^b(v) \otimes t^a(u)) d_{Y_i,Y_j}^{a,b} = d_{X,Y}^{a,b} (t^a(u) \otimes t^b(v)).$$

Indeed, the above relation is equivalent to the commutativity of the diagram below. Note that the last rectangle is commutative by Equation (1.9). Moreover, the other rectangles are commutative by the natural properties of the group action and Equation (1.1).

$$\begin{array}{ccc}
t^a(X) \otimes t^b(Y) & \xrightarrow{t^a(u) \otimes t^b(v)} & t^a(Y_i) \otimes t^b(Y_j) \\
\downarrow c(t^a(X), t^b(Y)) & & \downarrow c(t^a(Y_i), t^b(Y_j)) \\
t^{aga^{-1}}(t^b(Y)) \otimes t^a(X) & \dashrightarrow & t^{aga^{-1}}(t^b(Y_j)) \otimes t^a(Y_i) \\
\downarrow (t_2^{aga^{-1}, b})_Y \otimes 1 & & \downarrow (t_2^{aga^{-1}, b})_{Y_j} \otimes 1 \\
t^{aga^{-1}b}(Y) \otimes t^{ab}(X) & \dashrightarrow & t^{aga^{-1}b}(Y_j) \otimes t^{ab}(Y_i) \\
\downarrow (t_2^{b, b^{-1}aga^{-1}b})_Y^{-1} \otimes 1 & & \downarrow (t_2^{b, b^{-1}aga^{-1}b})_{Y_j}^{-1} \otimes 1 \\
t^b(t^{(b^{-1}a)g(b^{-1}a)^{-1}}(Y)) \otimes t^a(X) & \dashrightarrow & t^b(t^{(b^{-1}a)g(b^{-1}a)^{-1}}(Y_j)) \otimes t^a(Y_i) \\
\downarrow t^b(c_Y^{(b^{-1}a)g(b^{-1}a)^{-1}}) \otimes 1 & & \downarrow t^b(c_{Y_j}^{(b^{-1}a)g(b^{-1}a)^{-1}}) \otimes 1 \\
t^b(Y) \otimes t^a(X) & \xrightarrow{t^b(v) \otimes t^a(u)} & t^b(Y_j) \otimes t^a(Y_i).
\end{array}$$

Similarly one has that

$$(5.7) \quad (t^b(u) \otimes t^a(v))d_{Y_j, Y_i}^{b, a} = d_{Y, X}^{b, a}(t^a(v) \otimes t^b(u)).$$

Combining the above two equations one has that

$$(5.8) \quad (t^a(u) \otimes t^b(v))d_{Y_i, Y_j}^{a, b}d_{Y_j, Y_i}^{b, a} = d_{X, Y}^{a, b}d_{Y, X}^{b, a}(t^a(u) \otimes t^b(v))$$

Lemma 5.11. *With the above notations one has that*

$$(5.9) \quad t^a(d_{X, Y}^{b, c})(t_2^a)^{T^c(Y), T^b(X)} = (t_2^a)^{T^b(X), T^c(Y)}d_{X, Y}^{ab, ac}$$

Proof. Note that the identity is equivalent to the commutativity of the diagram below. In this diagram the first column represents $t^a(d_{X, Y}^{b, c})$ while the last column represents $d_{X, Y}^{ab, ac}$.

The above rectangle is commutative by the first condition (4.7) in the definition of a crossed braided fusion category. All the other squares are commutative by the properties of the group action by tensor equivalences.

$$\begin{array}{ccccc}
t^a(t^b(X) \otimes t^c(Y)) & \xrightarrow{(t^a)_2^{t^b(X), t^c(Y)}} & t^a(t^b(X)) \otimes t^a(t^c(Y)) & \xrightarrow{(t_2^{a,b})_X \otimes (t_2^{a,c})_Y} & t^{ab}(X) \otimes t^{ac}(Y) \\
\downarrow t^a(c(t^b(X) \otimes t^c(Y))) & & & & \downarrow c(t^{ab}(X), t^{ac}(Y)) \\
t^a(t^{bgb^{-1}}(t^c(Y)) \otimes t^b(X)) & \dashrightarrow & t^a(t^{bgb^{-1}}((t^c(Y)) \otimes t^b(X))) & \dashrightarrow & t^{abgb^{-1}a^{-1}}(t^{ac}(Y)) \otimes t^{ab}(X) \\
\downarrow t^a((t_2^{bgb^{-1},c})_Y \otimes 1) & & \downarrow & & \downarrow (t_2^{abgb^{-1}a^{-1},ac})_Y \otimes 1 \\
t^a(t^{bgb^{-1}c}(Y) \otimes t^b(X)) & \dashrightarrow & t^a(t^{bgb^{-1}c}(Y)) \otimes t^a(t^b(X)) & \dashrightarrow & t^{abgb^{-1}c}(Y) \otimes t^{ab}(X) \\
\downarrow t^a((t_2^{c,c^{-1}bgb^{-1}c})_Y \otimes 1) & & \downarrow & & \downarrow (t_2^{ac,c^{-1}bgb^{-1}c})_Y^{-1} \otimes 1 \\
t^a(t^c(t^{(b^{-1}c)^{-1}g(b^{-1}c)}(Y)) \otimes t^b(X)) & \dashrightarrow & t^a(t^c(t^{(b^{-1}c)^{-1}g(b^{-1}c)}(Y))) \otimes t^a(t^b(X)) & \dashrightarrow & t^{ac}(t^{c^{-1}bgb^{-1}c}(Y)) \otimes t^{ab}(X) \\
\downarrow t^a(t^c(c_Y^{(b^{-1}c)^{-1}g(b^{-1}c)}) \otimes 1) & & \downarrow & & \downarrow t^{ac}(c_Y^{c^{-1}bgb^{-1}c}) \otimes 1 \\
t^a(t^c(Y) \otimes t^b(X)) & \xrightarrow{(t^a)_2^{t^c(Y), t^b(X)}} & t^a(t^c(Y)) \otimes t^a(t^b(X)) & \xrightarrow{(t_2^{a,c})_Y \otimes (t_2^{a,b})_X} & t^{ac}(Y) \otimes t^{ab}(X)
\end{array}$$

□

Corollary 5.12. *With the above notations one has that*

$$(5.10) \quad (t_2^a)^{T^b(X), T^c(Y)} t^a(d_{X,Y}^{c,b} d_{X,Y}^{b,c}) = (d_{X,Y}^{ac,ab} d_{X,Y}^{ab,ac}) (t_2^a)^{T^b(X), T^c(Y)}$$

Proof. Applying Lemma 5.11 for $d_{X,Y}^{b,c}$ and its reverse $d_{Y,X}^{c,b}$ and then composing the two of them one obtains the above equation: □

Remark 5.13. Note that the above corollary implies that if $d_{X,Y}^{b,c} d_{Y,X}^{c,b} = \omega \text{id}_{t^b(X) \otimes t^c(Y)}$ then $d_{X,Y}^{ab,ac} d_{Y,X}^{ac,ab} = \omega \text{id}_{t^{ab}(X) \otimes t^{ac}(Y)}$.

Lemma 5.14. *With the above notation one has:*

$$(5.11) \quad t^a(d_{X,Y}^{b,c})((t_2^{ca^{-1},a} t_2^{a,c})_Y \otimes (t_2^{ba^{-1},a} t_2^{a,b})_X) = ((t_2^{ba^{-1},a} t_2^{a,b})_X \otimes (t_2^{ca^{-1},a} t_2^{a,c})_Y) d_{t^a(X), t^a(Y)}^{b,c}$$

Proof. The proof is similar to the proof of Lemma 5.11. It follows by a straightforward computation from Equation (4.8). □

As before, by composition this implies that

$$\begin{aligned}
& t^a[d_{X,Y}^{b,c} d_{Y,X}^{c,b}][t_2^{ba^{-1},a} t_2^{a,b})_X \otimes (t_2^{ca^{-1},a} t_2^{a,c})_Y] = \\
& = [(t_2^{ba^{-1},a} t_2^{a,b})_X \otimes (t_2^{ca^{-1},a} t_2^{a,c})_Y][d_{t^a(X), t^a(Y)}^{b,c} d_{t^a(Y), t^a(X)}^{c,b}]
\end{aligned}$$

Thus if $d_{X,Y}^{b,c} d_{Y,X}^{c,b} = \omega \text{id}_{t^b(X) \otimes t^c(Y)}$ for some scalar ω then

$$d_{t^a(X), t^a(Y)}^{b,c} d_{t^a(Y), t^a(X)}^{c,b} = \omega \text{id}_{t^b(t^a(X)) \otimes t^c(t^a(Y))}.$$

5.3. M\"uger's centralizer: necessary and sufficient conditions.

Next proposition generalizes [16, Lemma 5.1]. We keep the fixed system $C = \cup_Y C_Y$ of isomorphisms as above. Recall that we have chosen the systems of chosen isomorphisms C_{Y_i} such that the values of α_{Y_i} are all roots of unity. It follows that the values of $\alpha_{[Y]}$ are all roots of unity.

Proposition 5.15. *Suppose that $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ are two simple objects of a braided G -crossed fusion category. Then two simple objects $S_{X,\pi}$ and $S_{Y,\delta}$ centralize each other in \mathcal{C}^G if and only if the following conditions hold:*

- (1) K_g commutes elementwise with K_h .
- (2) $K_g \subseteq G_Y$ and $K_h \subseteq G_X$.
- (3) For all $m, n \in G$ there are some roots of unity $\omega_{X,Y}^{m,n} \in k^*$ such that the following two relations are satisfied:

$$(5.12) \quad d_{X,Y}^{m,n} d_{Y,X}^{n,m} = \omega_{X,Y}^{m,n} \text{id}_{t^m(X) \otimes t^n(Y)}$$

and

$$(5.13) \quad \pi(m^{-1}nhn^{-1}m) \otimes \delta(n^{-1}mgm^{-1}n) = (\omega_{X,Y}^{m,n})^{-1} \text{id}_{V_\pi \otimes V_\delta}.$$

Proof. Note that $G_X \subseteq C_G(g) \subseteq G$ where $g = \deg(X)$. Fix a set of representatives \mathcal{R}_g for the left cosets $G/C_G(g)$ and \mathcal{R}_X a set of representatives for the left cosets of $C_{G(g)}/G_X$. Recall that

$$S_{X,\pi} = \bigoplus_{m \in G/G_X} t^m(X) \otimes V_\pi = \bigoplus_{r \in \mathcal{R}_g, s \in \mathcal{R}_X} t^{rs}(X) \otimes V_\pi.$$

Similarly

$$S_{Y,\delta} = \bigoplus_{n \in G/G_Y} t^n(Y) \otimes V_\delta = \bigoplus_{a \in \mathcal{R}_h, b \in \mathcal{R}_Y} t^{ab}(Y) \otimes V_\delta.$$

Then one can apply Lemma 5.2 with $S = S_{X,\pi}$ and $T = S_{Y,\delta}$. In this situation one has $K_S = K_g$, $K_T = K_h$ with $S_{rgr^{-1}} = \bigoplus_{s \in \mathcal{R}_X} t^{rs}(X) \otimes V_\pi$ and $T_{aha^{-1}} = \bigoplus_{b \in \mathcal{R}_Y} t^{ab}(Y) \otimes V_\delta$.

Then Equation (5.4) for $u = rgr^{-1}$ and $v = aha^{-1}$ can be written as follows

$$\begin{aligned} \bigoplus_{s \in \mathcal{R}_X, b \in \mathcal{R}_Y} (d_{X,Y}^{rs,ab} d_{Y,X}^{ab,rs} \otimes (\pi((ab)g(ab)^{-1}) \otimes \delta((rs)h(rs)^{-1}))) &= \\ &= \bigoplus_{s \in \mathcal{R}_X, b \in \mathcal{R}_Y} \text{id}_{t^{rs}(X) \otimes t^{ab}(Y)} \otimes \text{id}_{V_\pi \otimes V_\delta}. \end{aligned}$$

Let $m := rs$ and $n := ab$. The last identity holds in the k -vector space

$$\begin{aligned} \text{End}_C(t^m(X) \otimes t^n(Y)) \otimes_k (\text{End}_k(V_\pi) \otimes \text{End}_k(V_\delta)) &\simeq \\ &\simeq \text{End}_C(t^m(X) \otimes t^n(Y)) \otimes_k \text{End}_k(V_\pi \otimes V_\delta). \end{aligned}$$

Note that one has $\text{End}_{\mathcal{C}}(X \otimes V) \simeq \text{End}_{\mathcal{C}}(X) \otimes \text{End}_k(V)$. This implies that there is some nonzero scalar $\omega_{m,n} \in k^*$ such that $d_{X,Y}^{m,n} d_{Y,X}^{n,m} = \omega_{m,n} \text{id}_{t^m(X) \otimes t^n(Y)}$ and $\pi(m^{-1}nhn^{-1}m) \otimes \delta(n^{-1}mgm^{-1}n) = \omega_{m,n}^{-1} \text{id}_{V_\pi \otimes V_\delta}$.

On the other hand, by the choice of the system C_X, C_Y it follows that if $\pi(g) = \alpha \text{id}_{V_\pi}$ then α must be a root of unity. This implies that all $\omega_{m,n}$ are roots of unity. \square

Corollary 5.16. *With the above notations the objects $S_{X,\pi}$ and $S_{Y,\delta}$ centralize each other if and only if for any $p \in G$ there is a root of unity $\omega_p(X, Y) \in k^*$ such that the following two relations hold:*

$$(5.14) \quad d_{X,Y}^{1,p} d_{Y,X}^{p,1} = \omega_p(X, Y) \text{id}_{X \otimes t^p(Y)}$$

$$(5.15) \quad \frac{\pi(php^{-1})}{\pi(1)} \otimes \frac{\delta(p^{-1}gp)}{\delta(1)} = \omega_p(X, Y)^{-1} \text{id}_{V_\pi \otimes V_\delta}$$

Proof. Remark 5.13 implies that if $X \perp_G Y$ then $\omega_{X,Y}^{m,n} = \omega_{X,Y}^{1,m^{-1}n} = \omega_{X,Y}^{n^{-1}m,1}$. The corollary follows by denoting $\omega_p(X, Y) := \omega_{X,Y}^{1,p}$ for any $p \in G$. \square

Remark 5.17. Suppose that G is an abelian group. Let $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ and suppose that the two simple objects $S_{X,\pi}$ and $S_{Y,\delta}$ centralize each other in \mathcal{C}^G . Then for any $m \in G$ the two identities above can be written as follows:

$$(5.16) \quad d_{X,Y}^{1,m} d_{Y,X}^{m,1} = \omega \text{id}_{X \otimes t^m(Y)}$$

and

$$(5.17) \quad \pi(h) \otimes \delta(g) = \omega^{-1} \text{id}_{V_\pi \otimes V_\delta}$$

Thus in this situation $S_{X,\pi}$ centralizes $S_{Y,\pi}$ if and only if $\deg(X) \in G_Y$, $\deg(Y) \in G_X$ and there is $\omega \in k^*$ root of unity such that the above two identities are satisfied.

5.4. Definition of the G -centralizer. Let \mathcal{C} be a braided G -crossed fusion category and two simple objects $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$. We say X and Y G -centralize each other (and we write $X \perp_G Y$) if there are projective representations $\pi \in \text{Irr}(k_{\alpha_X}[G_X])$ and $\delta \in \text{Irr}(k_{\alpha_Y}[G_Y])$ such that the simple objects $S_{X,\pi}$ and $S_{Y,\delta}$ centralize in \mathcal{C}^G . Equation (5.8) shows that if $X \perp_G Y$ and $X \simeq Y_i$, $Y \simeq Y_j$ then $Y_i \perp_G Y_j$. Moreover by the same relation we have that

$$(5.18) \quad \omega_{X,Y}^{m,n} = \omega_{Y_i,Y_j}^{m,n}$$

for all $m, n \in G$. For this reason we denote the above common value by $\omega_{[X],[Y]}^{m,n}$. Thus there is also a well defined quantity $\omega_p([X], [Y]) := \omega_{X,Y}^{1,p}$. For the rest of this paper we denote $\omega([X], [Y]) := \omega_1([X], [Y])$.

By Corollary 5.16 it follows that $X \perp_G Y$ if and only if there are roots of unity $\omega_p([X], [Y]) \in k^*$ and projective representations π, δ such that the above two Equations (5.14) and (5.15) are satisfied.

Note also that by its definition \perp_G is a symmetric relation, i.e. $X \perp_G Y$ if and only if $Y \perp_G X$. Moreover if $X \perp_G Y$ then $t^m(X) \perp_G t^n(Y)$ for all $m, n \in G$.

Remark 5.18. Note that if $X, Y \in \mathcal{C}_1$ then

$$d_{X,Y}^{m,n} = c_{t^m(X), t^n(Y)}$$

coincides to the braiding of \mathcal{C}_1 and does not depend on the chosen system of isomorphisms. Note that $d_{X,Y}^{m,n}$ depends only on c_Y^1 and c_X^1 which we supposed to be the identity. It follows that $X \perp_G Y$ if and only if $t^m(X)$ projectively centralize $t^n(Y)$ in \mathcal{C}_1 and there are projective representations π and δ such that for any $p \in G$ and Equations (5.14) and (5.15) are satisfied.

5.5. A formula for Müger centralizer. Recall that for a fusion datum $(H, \mathcal{S}, \lambda)$ as in Section 1 we denote by $\mathcal{C}(H, K, \mathcal{S}, \lambda)$ the corresponding fusion subcategory $\mathcal{C}(H, \mathcal{S}, \lambda)$ where K is the support of \mathcal{S} .

Proof of Theorem 1.4: Since H leaves invariant the objects of \mathcal{S} it follows that H commutes elementwise with K which proves i). Indeed, if $a \in K$, $b \in H$ and $X_a \in \mathcal{C}_a \cap \mathcal{S}$ then $X_a \cong t^b(X_a) \in \mathcal{C}_{bab^{-1}}$ which shows that $ab = ba$. Clearly $\mathcal{T} \perp_G \mathcal{S}$ by definition. The rest of the second item follows directly from Theorem 5.8.

For the next item we use Proposition 5.15. Suppose that $S_{X_a, \pi} \in \mathcal{C}(H, K, \mathcal{S}, \lambda)$ and $S_{Y_b, \delta} \in \mathcal{C}(K, H, \mathcal{T}, \tilde{\lambda})$. Then by Equation (3.5) one has $\lambda([X_a], b) = \frac{\chi_\pi(b)}{\chi_\pi(1)}$ and $\tilde{\lambda}([Y_b], a) = \frac{\chi_\delta(a)}{\chi_\delta(1)}$.

Since $S_{X_a, \pi}$ centralize $S_{Y_b, \delta}$ it follows by Equation (5.15) that for any $p \in G$ one has

$$\begin{aligned} \tilde{\lambda}([Y_b], pap^{-1}) &= \frac{\chi_\delta(p^{-1}ap)}{\chi_\delta(1)} = \left(\frac{\chi_\pi(pbp^{-1})}{\chi_\pi(1)} \right)^{-1} \omega_p^{-1}([X_a], [Y_b]) \\ &= \lambda^{-1}([X_a], pbp^{-1}) \omega_p^{-1}([X_a], [Y_b]) \end{aligned}$$

For $p = 1$ one obtains the desired equality. \square

Remark 5.19. Suppose that $S_{X, \pi} \in \mathcal{C}(H, K, \mathcal{S}, \lambda)$ with $X \in \mathcal{C}_a$. Since $\lambda([X], 1) = 1$ it follows from Equation (1.17) of the previous theorem that

$$(5.19) \quad \tilde{\lambda}([Y], a) = \omega([X], [Y])$$

for any $Y \in \mathcal{C}_1$ such that $S_{Y, \pi} \in \mathcal{C}(H, K, \mathcal{S}, \lambda)'$ and any $a \in K$.

Remark 5.20. Note that \mathcal{T} is contained in the largest fusion subcategory of \mathcal{C} whose objects G -centralize any object of \mathcal{S} .

6. BRAIDED G -CROSSED FUSION CATEGORIES FROM CROSSED G -MODULES

In this section we apply our results to equivariantizations of crossed pointed fusion categories obtaining a formula for the centralizer in any braided group theoretical category.

Recall that a fusion category \mathcal{C} is *pointed* if all simple objects of \mathcal{C} are invertible. Then there is an equivalence of fusion categories $\mathcal{C} \simeq \mathcal{C}(X, \omega)$, where X is the group of isomorphism classes of invertible objects in \mathcal{C} and $\omega : X \times X \times X \rightarrow k^*$ is an invertible normalized 3-cocycle. Moreover, $\mathcal{C}(X, \omega) = \text{Vec}_X^\omega$ is the category of finite dimensional X -graded vector spaces with associativity constraint induced by ω .

6.1. Group actions on $\mathcal{C}(X, \omega)$ and equivariantizations. (see [6])

Let $\mathcal{C} = \mathcal{C}(X, \omega)$ and let G be a finite group. An action $t : \underline{G} \rightarrow \underline{\text{Aut}}_\otimes \mathcal{C}$ of G on \mathcal{C} is determined by an action by group automorphisms of G on X , that we shall indicate by $x \mapsto {}^g x$, $x \in X$, $g \in G$, and two maps $\tau : G \times X \times X \rightarrow k^*$, and $\sigma : G \times G \times X \rightarrow k^*$, satisfying

$$\begin{aligned} \frac{\omega(x, y, z)}{\omega({}^g x, {}^g y, {}^g z)} &= \frac{\tau(g; xy, z) \tau(g; x, y)}{\tau(g; y, z) \tau(g; x, yz)} \\ 1 &= \frac{\sigma(h, l; x) \sigma(g, hl; x)}{\sigma(gh, l; x) \sigma(g, h; {}^l x)} \\ \frac{\tau(gh; x, y)}{\tau(g; {}^h x, {}^h y) \tau(h; x, y)} &= \frac{\sigma(g, h; x) \sigma(g, h; y)}{\sigma(g, h; xy)}, \end{aligned}$$

for all $x, y, z \in X$, $g, h, l \in G$.

We shall also assume that τ and σ satisfy the additional normalization conditions $\tau(g; x, y) = \sigma(g, h; x) = 1$, whenever some of the arguments g, h, x or y is an identity.

The action $t : \underline{G} \rightarrow \underline{\text{Aut}}_\otimes \mathcal{C}$ determined by this data is defined by letting $t^g(x) = {}^g x$, for all $g \in G$, $x \in X$, and $t^g = \text{id}$ on arrows, together with the following constraints:

$$(6.1) \quad (t_2^{g,h})_x = \sigma(g, h; x)^{-1} \text{id}_{g^h x}, \quad (t_2^g)^{x,y} = \tau(g; x, y)^{-1} \text{id}_{xy}, \quad t_0^g = \text{id}_e,$$

for all $g, h \in G$, $x, y \in X$. See also [21, Section 7].

Recall that a fusion category is called *group-theoretical* if it is Morita equivalent to a pointed fusion category. By [16, Theorem 7.2], a *braided* fusion category is group-theoretical if and only if it is an equivariantization of a pointed fusion category. Moreover, it was shown in [17, Theorem 5.3] that every braided group-theoretical fusion category is equivalent to an equivariantization $\mathcal{C}(\xi)^G$ of a crossed pointed fusion category $\mathcal{C}(\xi)$ associated to a quasi-abelian 3-cocycle ξ on a finite crossed module (G, X, ∂) , under a canonical action of G on $\mathcal{C}(\xi)$.

Recall that a finite crossed module is a triple (G, X, ∂) consisting of a finite group G acting by automorphisms on a finite group X , and a group homomorphism $\partial : X \rightarrow G$ such that

$$\partial(x)y = xyx^{-1}, \quad \partial(gx) = g\partial(x)g^{-1}, \quad g \in G, \quad x, y \in X,$$

where $x \mapsto^g x$, $x \in X$, $g \in G$, denotes the action of g on X .

As a fusion category $\mathcal{C}(\xi) = \mathcal{C}(X, \omega)$, and the action of G on $\mathcal{C}(\xi)$ is determined by the action of G on X and formulas (6.1), with respect to $\sigma(g, h; x) := \gamma(g, h; x)^{-1}$, $\tau(g, x, y) := \mu(g; x, y)^{-1}$, $x, y \in X$, $g, h \in G$. See [17, Subsection 4.1]. Following loc. cit. we will use the notation $\gamma_{g,h}(x) := \gamma(g, h; x)$.

Recall that a quasi-abelian 3-cocycle ξ on (G, X, ∂) is a quadruple $\xi = (\omega, \gamma, \mu, c)$, where $\omega : X \times X \times X \rightarrow k^*$ is a 3-cocycle, $\gamma : G \times G \times X \rightarrow k^*$, $\mu : G \times X \times X \rightarrow k^*$ and $c : X \times X \rightarrow k^*$ are maps satisfying the compatibility conditions in [17, Definition 3.4].

6.2. Simple objects of the equivariantized category $\mathcal{C}(\omega)^G$. Fix the system $C \cong 1$ of chosen isomorphisms given by $c_x^g : k_{gx} \rightarrow k_x = \text{id}_{k_x}$ as identity, for any $g \in G_x$, the stabilizer group of x . Fix also a set Γ of representatives for the orbits of the action.

Using Equation (1.6) it follows for any $a \in \Gamma$ one has that

$$(6.2) \quad \alpha_x(g, h) = \gamma_{g,h}(x)$$

for any $x \in X$ and any $g, h \in G$.

Following Equation (2.4) it follows that

$$(6.3) \quad D_{g,y}(h) = \frac{\gamma_{ghg^{-1},g}(y)}{\gamma_{g,h}(y)}$$

Note that the projective representation $\tau_{yz}^{y,z}$ is one dimensional given by

$$g.1_k = \tau(g; y, z)^{-1}1_k.$$

Let π be a α_x -projective representation of the stabilizer G_x . By definition $S_{x,\pi}$ is given by:

$$(6.4) \quad S_{x,\pi} = \bigoplus_{t \in G/G_x} \mathfrak{t}^t(x) \otimes V_\pi$$

The equivariant structure of $S_{x,\pi}$ is given on the components by

$$(6.5) \quad \mathfrak{t}^g(\mathfrak{t}^t(a)) \otimes v \xrightarrow{\gamma_{g,t}(x) \otimes 1} \mathfrak{t}^{gt}(x) \otimes v \xrightarrow{\gamma_{t',h}^{-1}(x) \otimes \pi(h)} \mathfrak{t}^{t'}(x) \otimes hv$$

where t' is chosen such that $gt = t'h$ with $h \in G_x$. It can be easily seen that this coincides to the simple module denoted by (a, π) in [17].

6.3. G grading on \mathcal{C} . For each $g \in G$, let \mathcal{C}_g denote the full abelian subcategory consisting of objects of Vec_X supported on $\delta^{-1}(g) \subset X$, i.e., objects of \mathcal{C}_g are defined to be finite-dimensional $\delta^{-1}(g)$ -graded vector spaces.

Recall that the G -braiding \tilde{c} is defined by

$$(6.6) \quad c(x, y) \text{id}_{k_{(g_x)y}} = k_x \otimes k_y \rightarrow k_{g_x} \otimes k_y,$$

where $g = \delta(x)$.

6.4. Fusion subcategories of \mathcal{C}^G . By Definition 1.2 all fusion subcategories of \mathcal{C}^G are parameterized by triples (H, Y, λ) where H is a normal subgroup of G , Y a subgroup of X which is stable under the action of G and fixed by H . Moreover $\lambda : H \times Y \rightarrow k^*$ is a twisted bicharacter satisfying the following relations:

$$\begin{aligned} \lambda(y, hh') &= \lambda(y, h)\lambda(y', h)\gamma_{h,h'}(y) \\ \lambda(yz, h) &= \lambda(y, h)\lambda(z, h)\tau(h; y, z) \\ \lambda({}^g y, h) &= \lambda(y, g^{-1}hg)D_{g,y}(h) \end{aligned}$$

for any $y, z \in Y$, any $g \in G$ and any $h \in H$.

6.5. The double distance in X and the G -centralizer. Suppose that we have two objects k_x, k_y for which their conjugacy classes in G of their degrees commute element by element. Then the distance from subsection 5.2 is given by

$$(6.7) \quad d_{x,y}^{m,n} : k_{m_x} \otimes k_{n_y} \rightarrow k_{n_y} \otimes k_{m_x}$$

is given by multiplication with the scalar

$$(6.8) \quad \gamma_{mgm^{-1},n}(y)\gamma_{n,n^{-1}mgm^{-1}n}^{-1}(y)c_{m_x, n_y}c_{m_x, n_y}$$

where $g = \deg(x)$. In particular $d_{x,y}^{1,n} = \gamma_{g,n}(y)\gamma^{-1}(n, n^{-1}gn)$

Thus for any two elements $x, y \in X$ such that the conjugacy classes in G of $\delta(x)$ and $\delta(y)$ commute each other the elements k_x and k_y of \mathcal{C}^G centralize each other.

Thus by Equation (5.16) one has that

$$(6.9) \quad \omega(x, y) = d_{x,y}^{1,n} d_{y,x}^{n,1} = \gamma_{g,n}(y) \gamma^{-1}(n, n^{-1}gn) c_{x, {}^n y} \gamma_{nhn^{-1},1}(x) \gamma_{1,h}^{-1}(x) c_{{}^n y, x}$$

where $h = \deg(y)$.

6.6. M\"uger centralizer formula for equivariantizations of cross pointed fusion categories. Applying Theorem 1.4 one obtains the following:

Theorem 6.1. *Suppose that δ is surjective and (ω, γ, μ, c) is nondegenerate in the sense of [17, Definition 3.10]. Then for any fusion subcategory one has the following*

$$(6.10) \quad \mathcal{C}(H, Y, \lambda)' = \mathcal{C}(\delta(Y), T, \tilde{\lambda})$$

where T is a subgroup of X commuting with Y and with $\delta(T) = H$. Moreover in this case one has that:

$$\tilde{\lambda}(\delta(y), t) = \lambda^{-1}(h, y) \omega(y, t)$$

for any $h \in H$ and $y \in Y$ and $t \in T$ with $\delta(t) = h$.

Proof. Following [17, Proposition 5.6] we deduce that $\mathcal{C}(\xi)^G$ is nondegenerate and we can apply Theorem 1.4. \square

On the other hand from the non-degeneracy condition it follows by Equation (4.5) that

$$(6.11) \quad \text{FPdim}(\mathcal{C}(H, Y, \lambda)') = \frac{\text{FPdim}(\mathcal{C}^G)}{\text{FPdim}(\mathcal{C}(H, Y, \lambda))} = \frac{|G||X|}{|G||Y|/|H|} = \frac{|X||H|}{|Y|}$$

One has that $\mathcal{C}(H, Y, \lambda)' = \mathcal{C}(\delta(Y), \mathcal{T}, \lambda')$ where the support $\delta(\mathcal{T})$ of \mathcal{T} coincides to H . Thus $\mathcal{T} \subseteq \delta^{-1}(H)$.

On the other hand using Equation (3.8) one has that

$$(6.12) \quad \text{FPdim}(\mathcal{C}(\delta(Y), \mathcal{T}, \tilde{\lambda})) = \frac{|G||T|}{|\delta(Y)|}$$

Note that $|\delta(Y)| = \frac{|Y|}{|\ker(\delta) \cap Y|}$ and $|\ker(\delta)||G| = |X|$. Then the last two equations give that

$$(6.13) \quad |T| = |H| \frac{|\ker(\delta)|}{|\ker(\delta) \cap Y|}.$$

Thus choosing a subgroup Y such that $Y \cap \ker \delta \neq 1$ it follows that \mathcal{T} is not the largest subcategory mentioned above in Remark 5.20.

As explained in the last paragraph of [17] the twisted Drinfeld double $D^\omega(G)$ of a finite group can be recovered as an equivariantization of a crossed pointed fusion category. Thus the last theorem generalizes the results obtained in [16, Theorem 5.11].

7. APPENDIX

In this Appendix we give a brief account of the results on projective representations used throughout this paper. See for instance [14]. Most of the facts of this Appendix are also contained in the Appendix of [6].

Let G be a finite group and let $\alpha : G \times G \rightarrow k^*$ be a (normalized) 2-cocycle on G , that is,

$$\alpha(g, h)\alpha(gh, t) = \alpha(g, ht)\alpha(h, t), \quad \alpha(g, e) = 1 = \alpha(e, g), \quad \forall g, h, t \in G.$$

A α -projective representation π of G with associated cocycle α on a vector space V is a map $\pi : G \rightarrow \text{GL}(V)$, such that

$$\pi(e) = \text{id}_V, \quad \pi(gh) = \alpha(g, h)\pi(g)\pi(h), \quad \forall g, h \in G.$$

In other words, π is a representation of the twisted group algebra $k_\alpha G$ on the vector space V . We shall also use the notation $V_\pi = V$ to indicate such a projective representation.

Two projective representations π and π' of G are called (*projectively*) *equivalent* if there is a linear isomorphism $\phi : V_\pi \rightarrow V_{\pi'}$ and a map $f : G \rightarrow k^*$ such that $\phi\pi(g) = f(g)\pi'(g)\phi$, for all $g \in G$. In this case we shall use the notation $\pi' \simeq \pi$.

If $\pi' \simeq \pi$, then the associated cocycles α and α' are related by

$$\alpha(g, h) = \alpha'(g, h)f(g)f(h)f(gh)^{-1}, \quad g, h \in G,$$

that is, α and α' are cohomologous cocycles, and thus they belong to the same cohomology class $\alpha \in H^2(G, k^*)$. Note that the map $f : G \rightarrow k^*$ induces an algebra isomorphism $\tilde{f} : k_\alpha G \rightarrow k_{\alpha'} G$ in the form $\tilde{f}(g) = f(g)g$, for all $g \in G$. Thus π and π' are equivalent projective representations if and only if $V_\pi \simeq \tilde{f}^*(V_{\pi'})$ as $k_{\tilde{\alpha}} G$ -modules.

Remark 7.1. (see [14]) Given a two cocycle $\alpha : G \times G \rightarrow k^*$ one can always find a cohomologous cocycle α' such that the values $\alpha'(g, g')$ are all roots of unity for all $g, g' \in G$.

Let π and π' be projective representations of G with associated cocycles α and α' , respectively. The *tensor product* $\pi \otimes \pi'$ is the $\alpha\alpha'$ -projective representation of G on the vector space $V_\pi \otimes V_{\pi'}$ defined by $(\pi \otimes \pi')(g)(u \otimes v) = \pi(g)u \otimes \pi'(g)v$. In particular, if π is a representation of G , then $\pi \otimes \pi'$ is again a α' -projective representation with associated cocycle α' .

If π_1 and π'_1 are projective representations projectively equivalent to π and π' , respectively, then the tensor products $\pi_1 \otimes \pi'_1$ and $\pi \otimes \pi'$ are projectively equivalent. Further, suppose that π' is a one-dimensional representation, that is, a linear character of G . Then π and $\pi \otimes \pi'$ are projectively equivalent via the canonical isomorphism $\phi : V_\pi \rightarrow V_\pi \otimes k$, $v \mapsto v \otimes 1$, and the map $f : G \rightarrow k^*$ given by $f(g) = \pi'(g)^{-1}$, for all $g \in G$.

A nonzero projective representation $\pi : G \rightarrow \text{GL}(V)$ of G is called *irreducible* if 0 and V are the only subspaces of V which are invariant under $\pi(g)$, for all $g \in G$. Hence, π is irreducible if and only if it is not projectively equivalent to a projective representation ρ of the form

$$\rho(g) = \begin{pmatrix} \pi_1(g) & * \\ 0 & \pi_2(g) \end{pmatrix}, \quad g \in G,$$

where π_1 and π_2 are nonzero projective representations or, equivalently, if V is a simple $k_{\tilde{\alpha}}G$ -module, where $\tilde{\alpha}$ is the factor set of π [14, Theorem 3.2.5].

Let $\pi : G \rightarrow \text{GL}(V)$ be a projective representation of G with factor set $\tilde{\alpha}$. Since the group algebra $k_{\tilde{\alpha}}G$ is semisimple, then $V = V_\pi$ is *completely reducible*, that is, $V_\pi \simeq V_{\pi_1} \oplus \cdots \oplus V_{\pi_n}$, where V_{π_i} is a simple $k_{\tilde{\alpha}}G$ -module, for all $i = 1, \dots, n$. If π' is an irreducible projective representation with factor set $\tilde{\alpha}'$, then π' is called a *constituent* of π if π' is projectively equivalent to π_i for some $1 \leq i \leq n$. In this case, the *multiplicity* (or *intertwining number*) of π' in π is defined as

$$m_G(\pi', \pi) := \dim \text{Hom}_{k_{\tilde{\alpha}}G}(V_{\pi_i}, V_\pi).$$

Observe that if π' is a constituent of π , then the cocycles $\tilde{\alpha}'$ and $\tilde{\alpha}$ belong to the same class in $H^2(G, k^*)$. Letting $\tilde{\alpha}'df = \tilde{\alpha}$, with $f : G \rightarrow k^*$, we have that $m_G(\pi', \pi) := \dim \text{Hom}_{k_{\tilde{\alpha}}G}(f^*(V_{\pi'}), V_\pi)$, where $\tilde{f} : k_{\tilde{\alpha}}G \rightarrow k_{\tilde{\alpha}'}G$ is the isomorphism associated to f .

The *character* of a projective representation $\pi : G \rightarrow \text{GL}(V)$ is defined as the map $\chi = \chi_V : G \rightarrow k^*$ given by $\chi(g) = \text{Tr}(\pi(g))$, for all $g \in G$. Let $\tilde{\alpha}$ be the factor set of π . If π' is an irreducible projective representation of G with factor set $\tilde{\alpha}$ and character χ' , then the multiplicity of π' in π can be computed by the formula

$$\begin{aligned} m_G(\pi', \pi) &= \langle \chi', \chi \rangle := \frac{1}{|G|} \sum_{g \in G} \frac{1}{\alpha(g^{-1}, g)} \chi'(g) \chi(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G^0} \frac{1}{\alpha(g^{-1}, g)} \chi'(g) \chi(g^{-1}), \end{aligned}$$

$G^0 \subseteq G$ is the subset of $\tilde{\alpha}$ -regular elements of G . See [14, Chapter 5].

Let $\alpha : G \times G \rightarrow k^*$ be a 2-cocycle and let $H \subseteq G$ be a subgroup. Consider a projective representation W of H with 2-cocycle $\alpha|_H$. The *induced projective representation* of G is defined as $\text{Ind}_H^G W = k_\alpha G \otimes_{k_\alpha H} W$. This is a projective representation of G with factor set α . By Frobenius reciprocity, we have natural isomorphisms

$$(7.1) \quad \text{Hom}_{k_\alpha G}(\text{Ind}_H^G W, V) \simeq \text{Hom}_{k_\alpha H}(W, V|_H),$$

for every projective representation V of G with factor set α , where $V|_H$ denotes the restricted projective representation of H .

7.1. On the center of a projective representation. Let M be a α -projective G -representation with associated cocycle α . Define

$$(7.2) \quad Z_\alpha(M) := \{g \in G \mid gm = \psi(g)m \text{ for some } \psi(g) \in k^* \text{ for all } m \in M\}.$$

Clearly $Z_\alpha(M)$ is a subgroup of G and $\psi_M := \psi$ defines a one dimensional representation of $k_{\alpha|_{Z_\alpha(M)}}[Z_\alpha(M)]$.

Lemma 7.2. *Let π be a α -projective representation of G with $\alpha(g, g')$ all roots of unity. Then for $h \in G$ one has that $|\chi_\pi(h)| \leq \chi_\pi(1)$ and one has equality if and only if $h \in Z_\alpha(\pi)$.*

Proof. Since all the values $\alpha(h^i, h^j)$ are all roots of unit it follows that there is $n > 0$ such that $h^n = 1$ in $k_\alpha[H]$. Therefore $\chi_\pi(h)$ is a sum of roots of unity and the statement follows. \square

Lemma 7.3. *Let $H \subset T \subset G$ be a tower of finite groups and π be α -projective representation of G . If $H \subset Z_{\alpha|_H}(\pi \uparrow_T^G)$ then $H \subset Z_\alpha(\pi)$.*

Proof. Using Equation (7.1) it can be easily checked that π is a constituent of $\pi \uparrow_T^G \downarrow_T^G$. \square

It is straightforward to check the following lemma.

Lemma 7.4. *Let M be a subgroup of G and γ be the character of a representation of M . Then $\ker_G(\gamma \uparrow_M^G) = \text{core}_G(\ker_M(\gamma))$.*

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